On the Power and Limitations of Branch and Cut

Noah Fleming

Joint work with
Göös, Tan, Impagliazzo, Pitassi, Robere, Wigderson

In This Talk...

- Algorithm analysis from proof complexity
- The proof complexity of integer programming
  - Branch-and-cut & Stabbing Planes
- Cutting Planes vs Stabbing Planes
- Short proofs of $\mathbb{F}_q$ linear equations
- Deep Cutting Planes Proofs

Algorithm Analysis from Proofs

Idea: Formalize the techniques used in a class of algorithms A as a proof system P

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- ▷ Hides practical details of algorithms
- ▷ Lower bounds on P-proofs → lower bounds on runtime of A

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- e.g. Algorithms for SAT
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- e.g. Algorithms for Integer Programming
  - Chvátal-Gomory Cutting Planes and Cutting Planes

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Integer-programming: Given $Ax \geq b$ find $x \in \mathbb{Z}^n$, $Ax \geq b$

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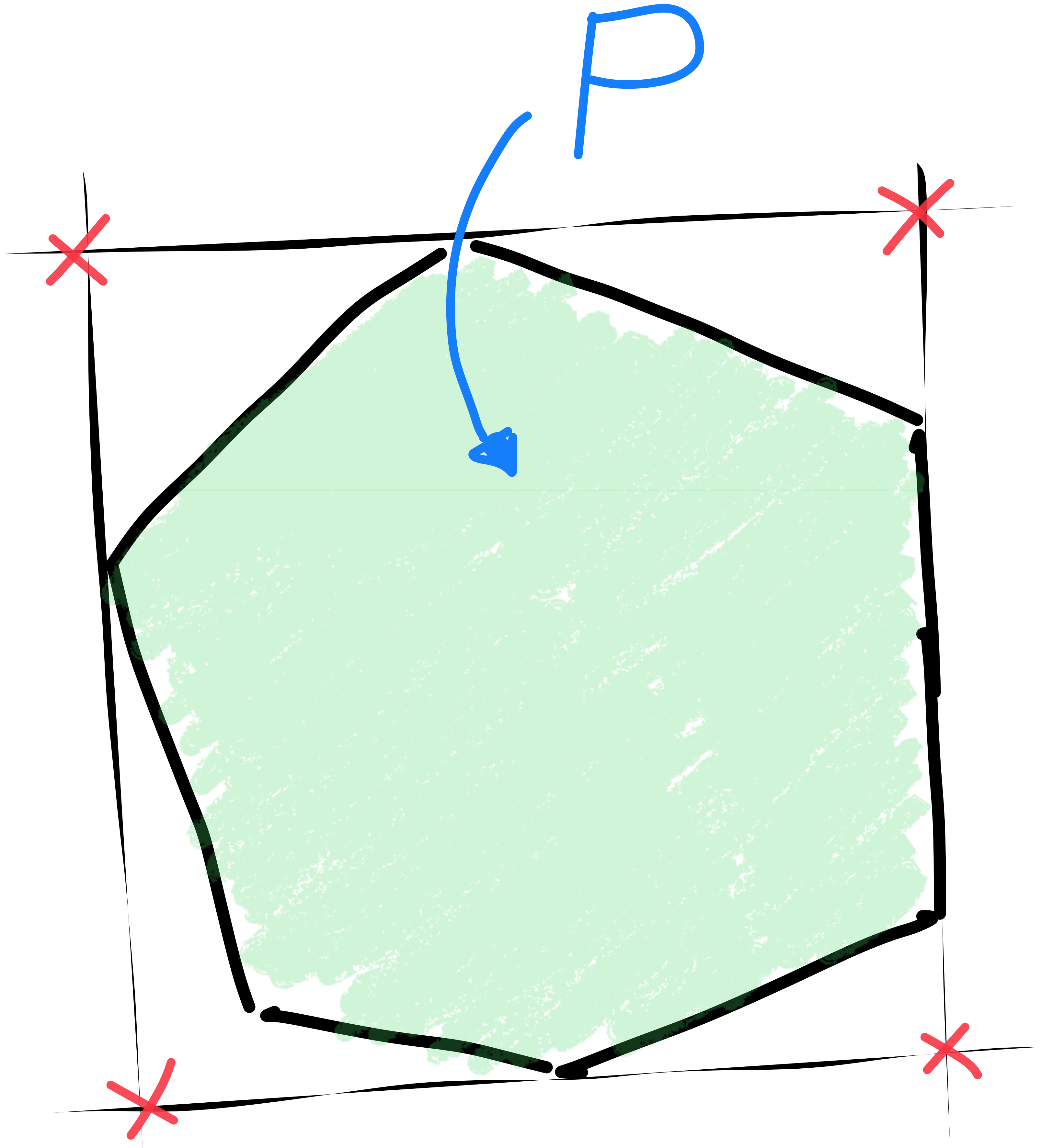

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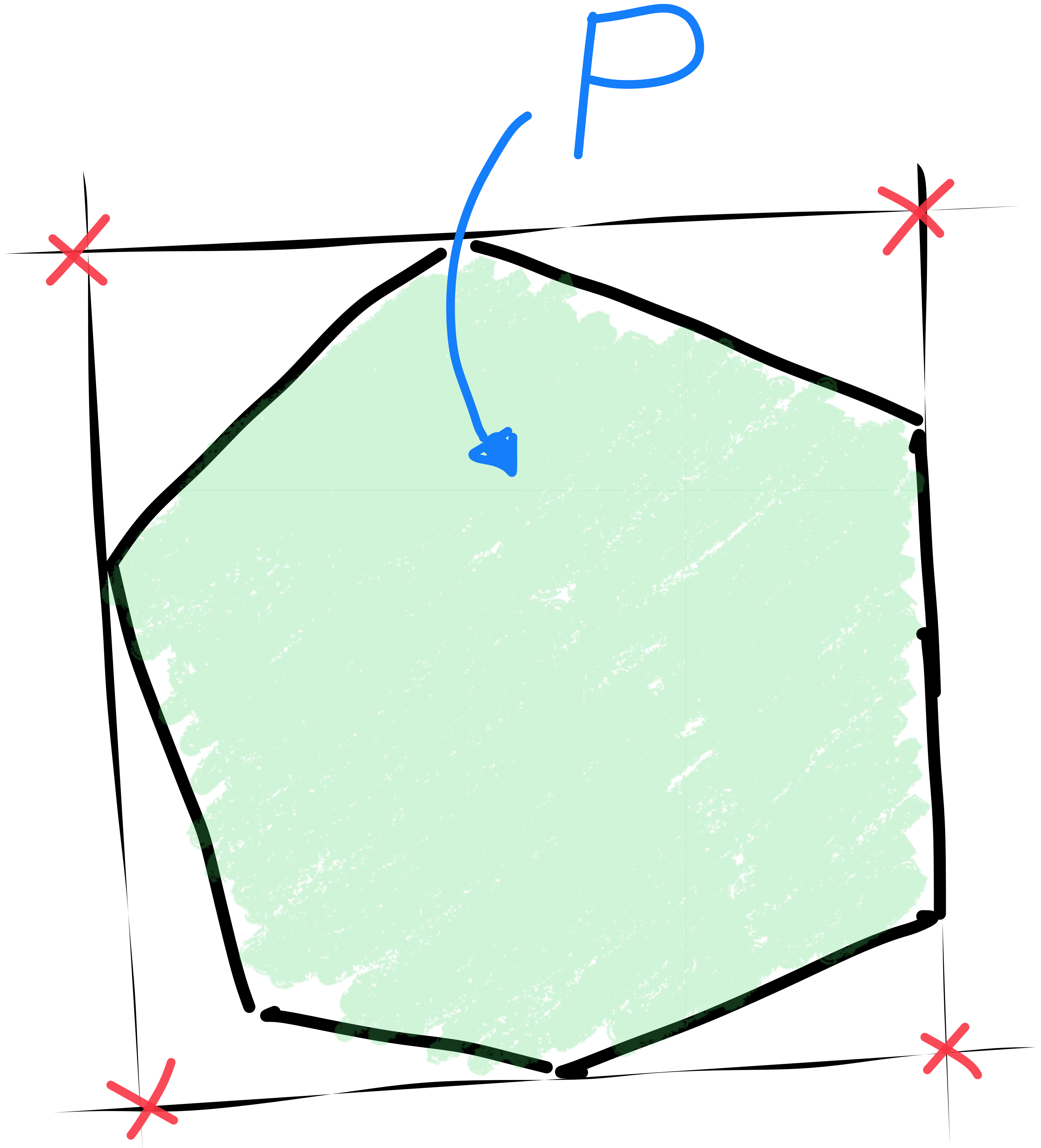
A Classic approach: Chvátal-Gomory Cutting Planes



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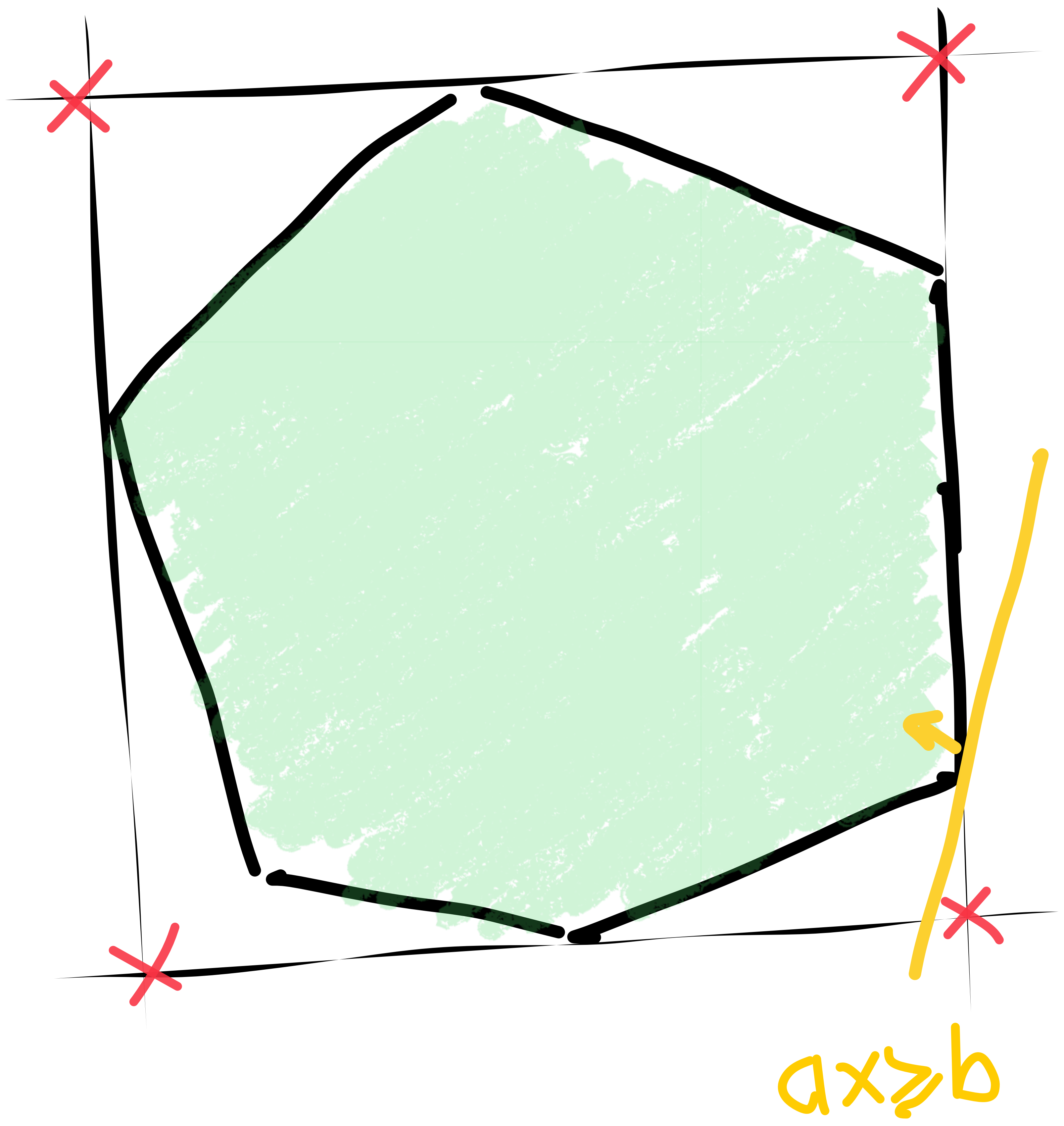

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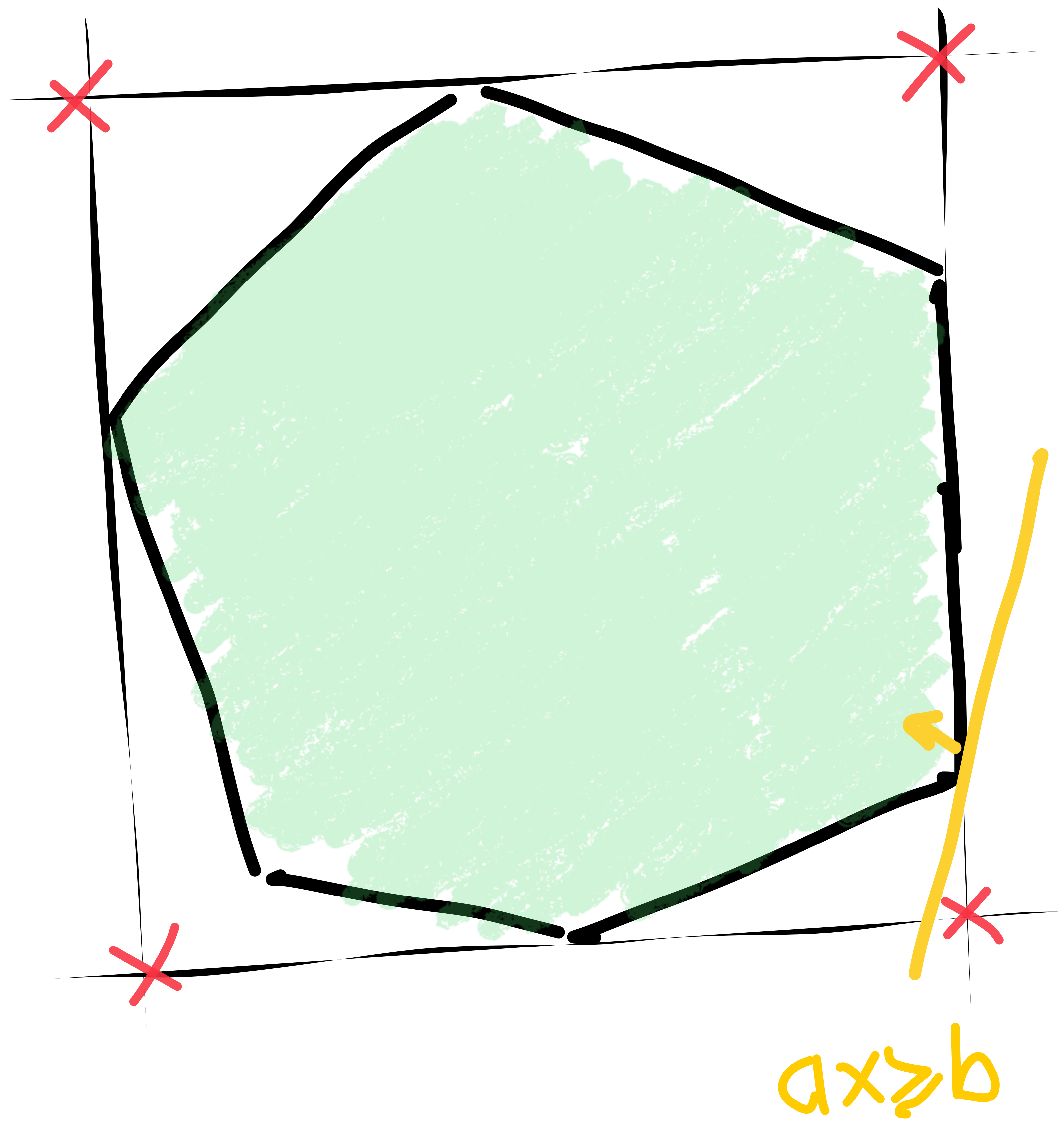

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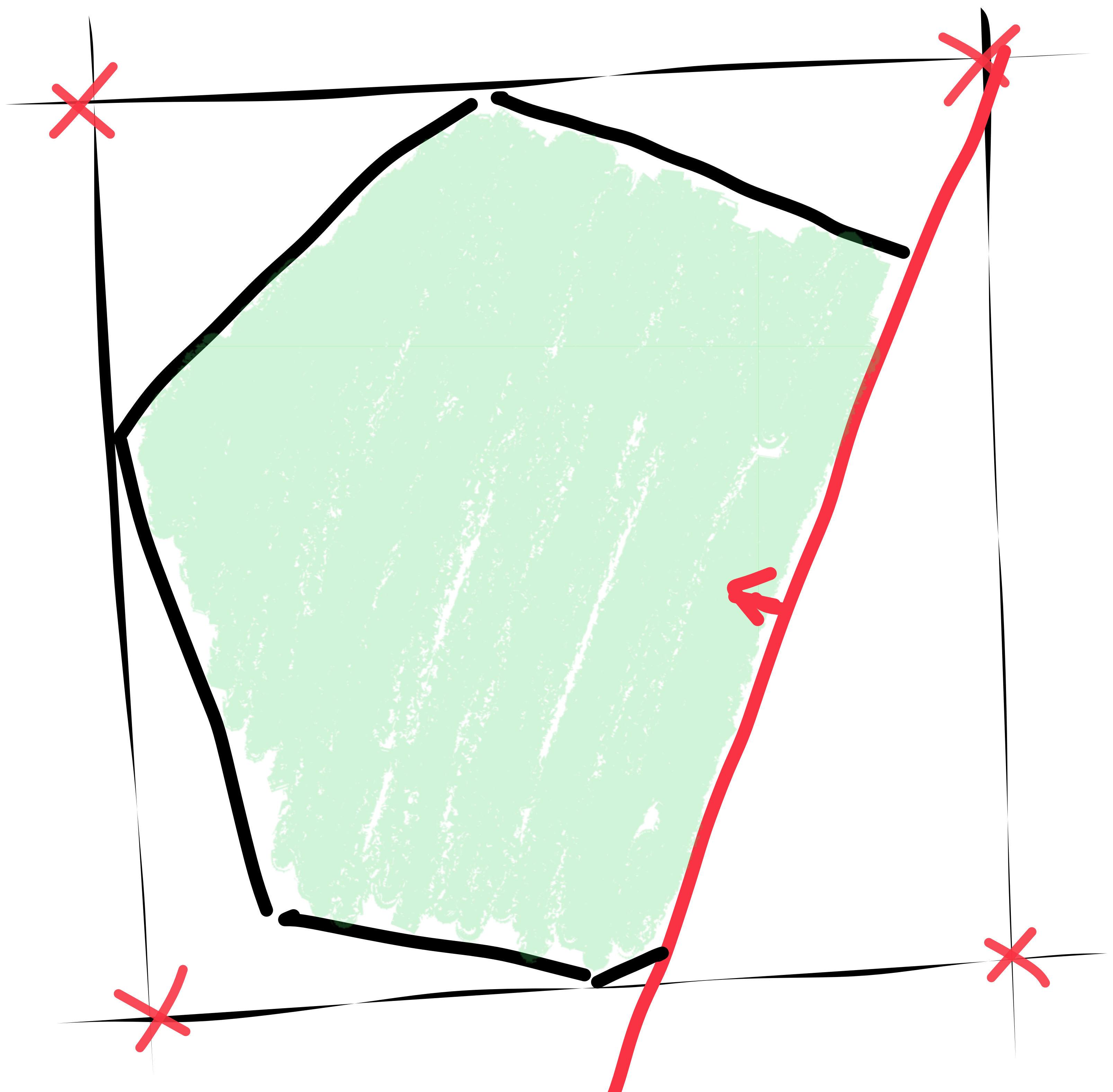

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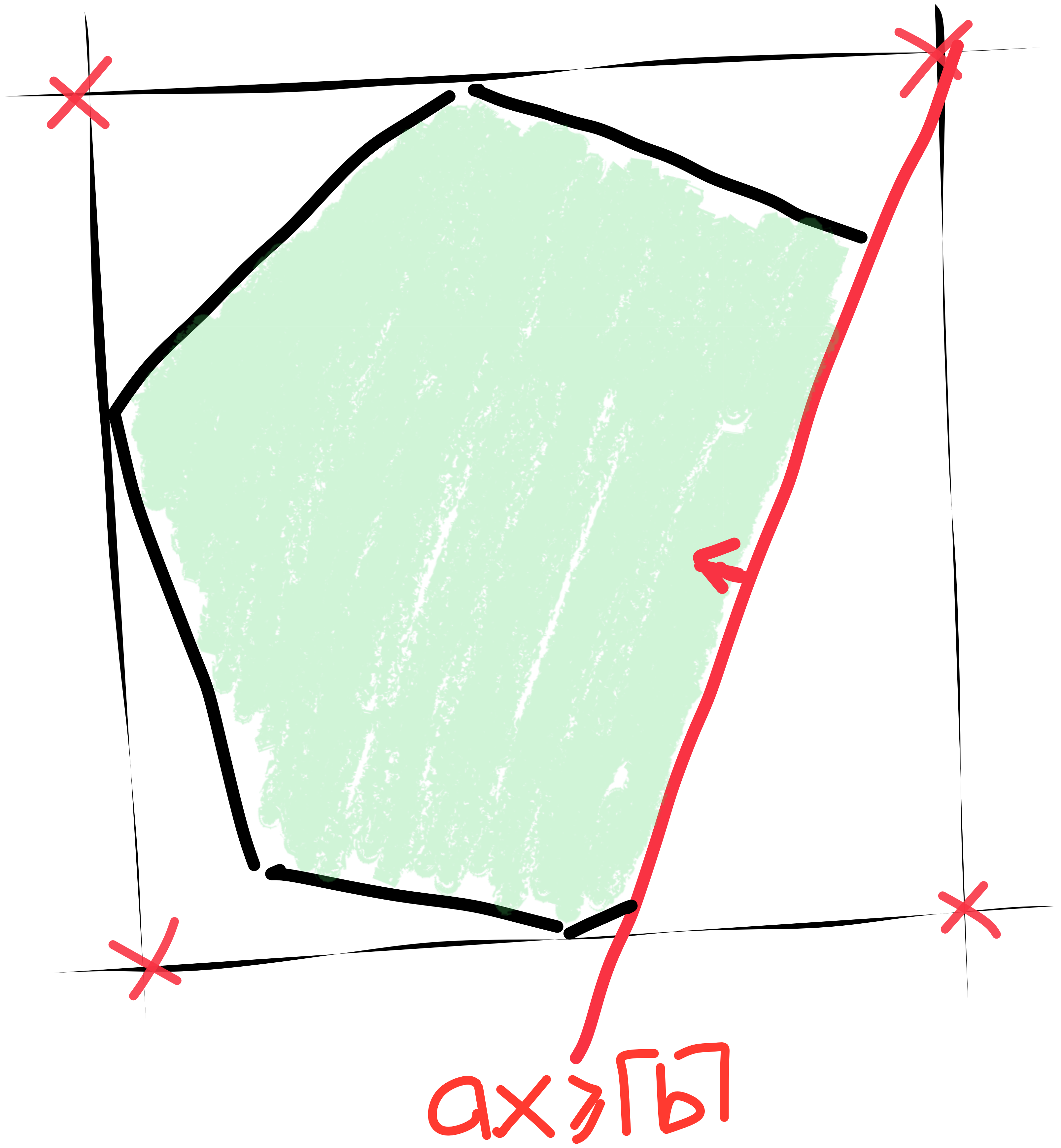

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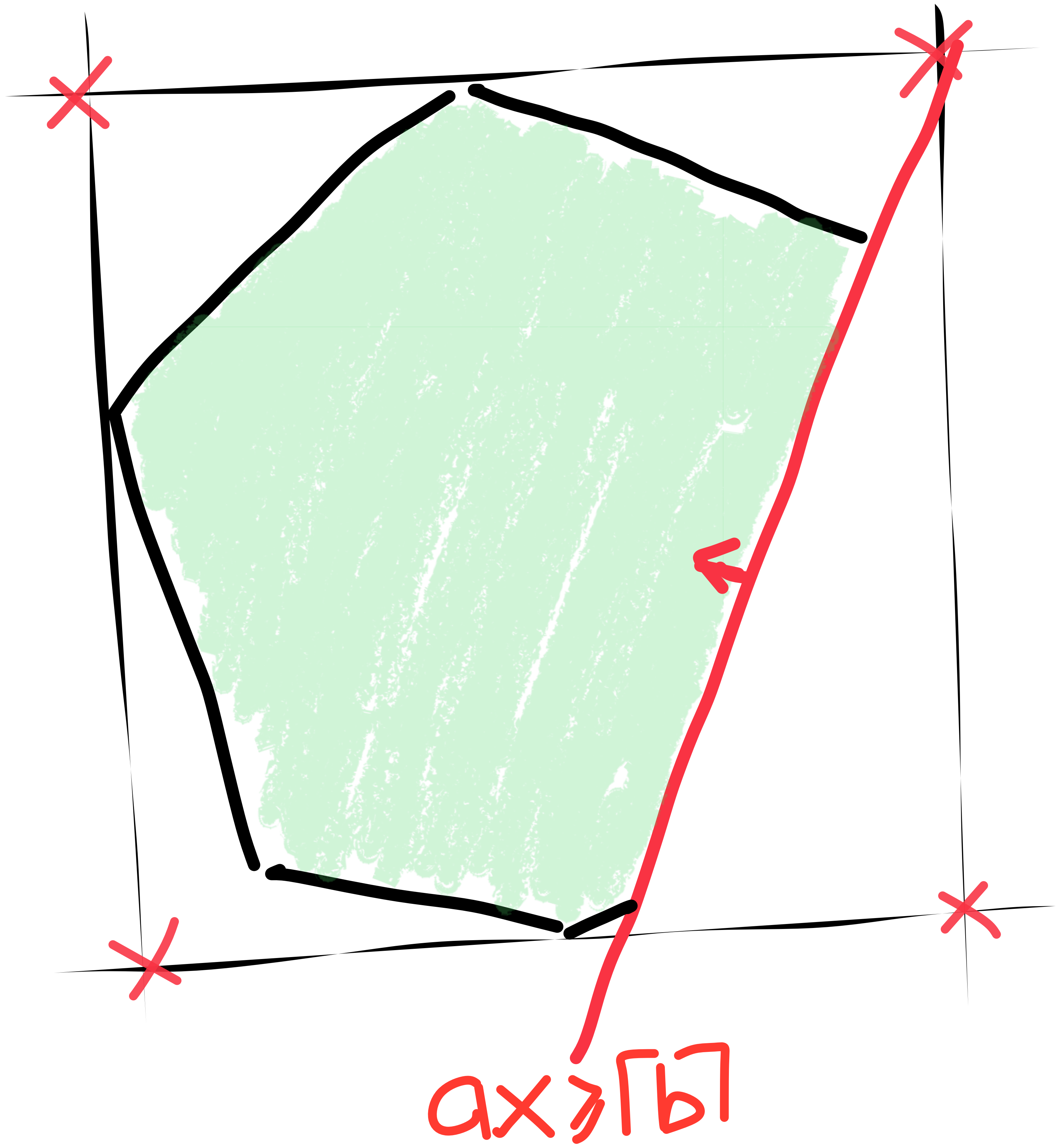

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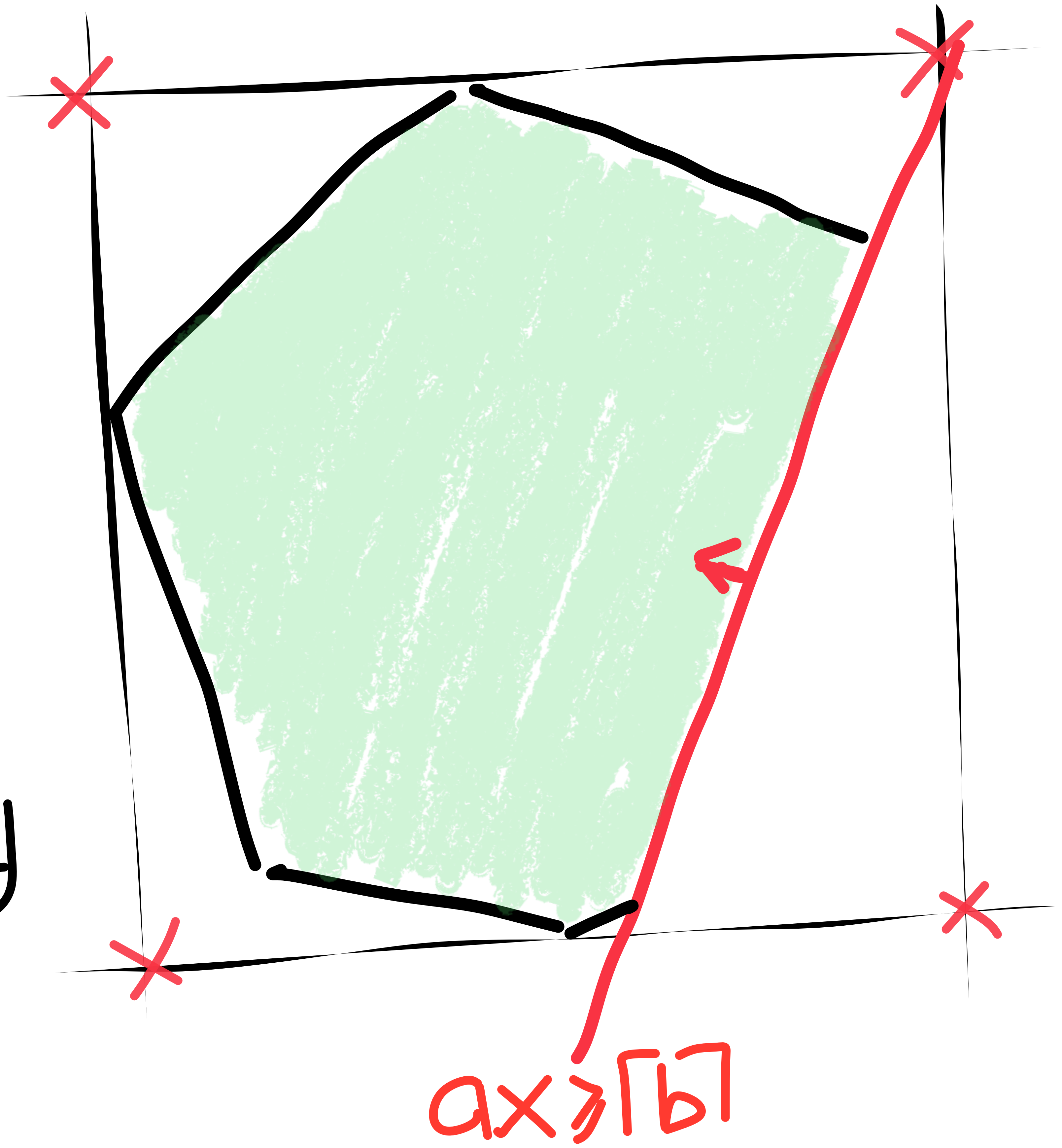
CG-Cutting Planes

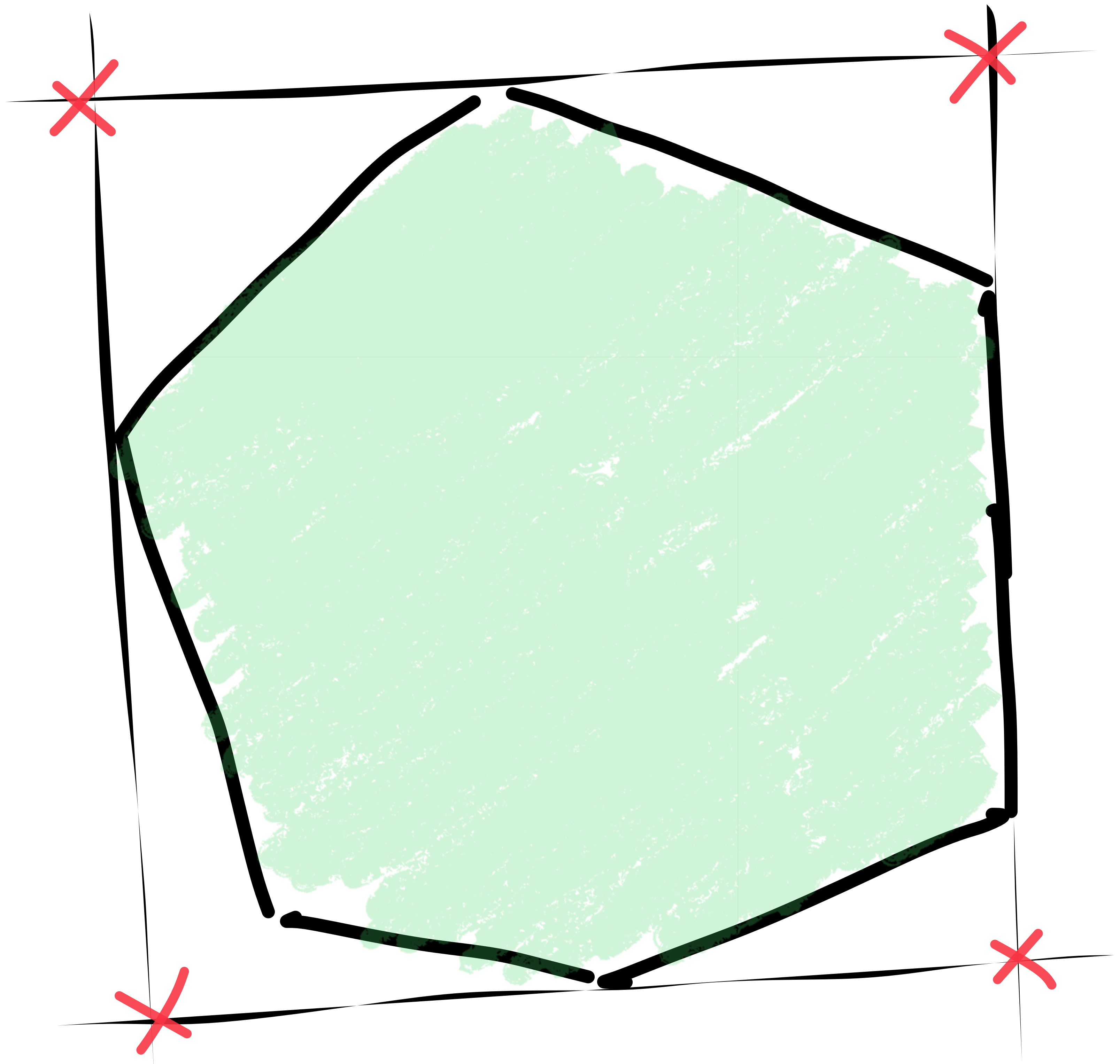
Heuristically add CG-cuts to P until:

- an integer solution is found
- the empty polytope is deduced



Cutting Planes [CCT87]

Let $P = \{Ax \geq b\}$ be such that $P \cap \mathbb{Z}^n = \emptyset$



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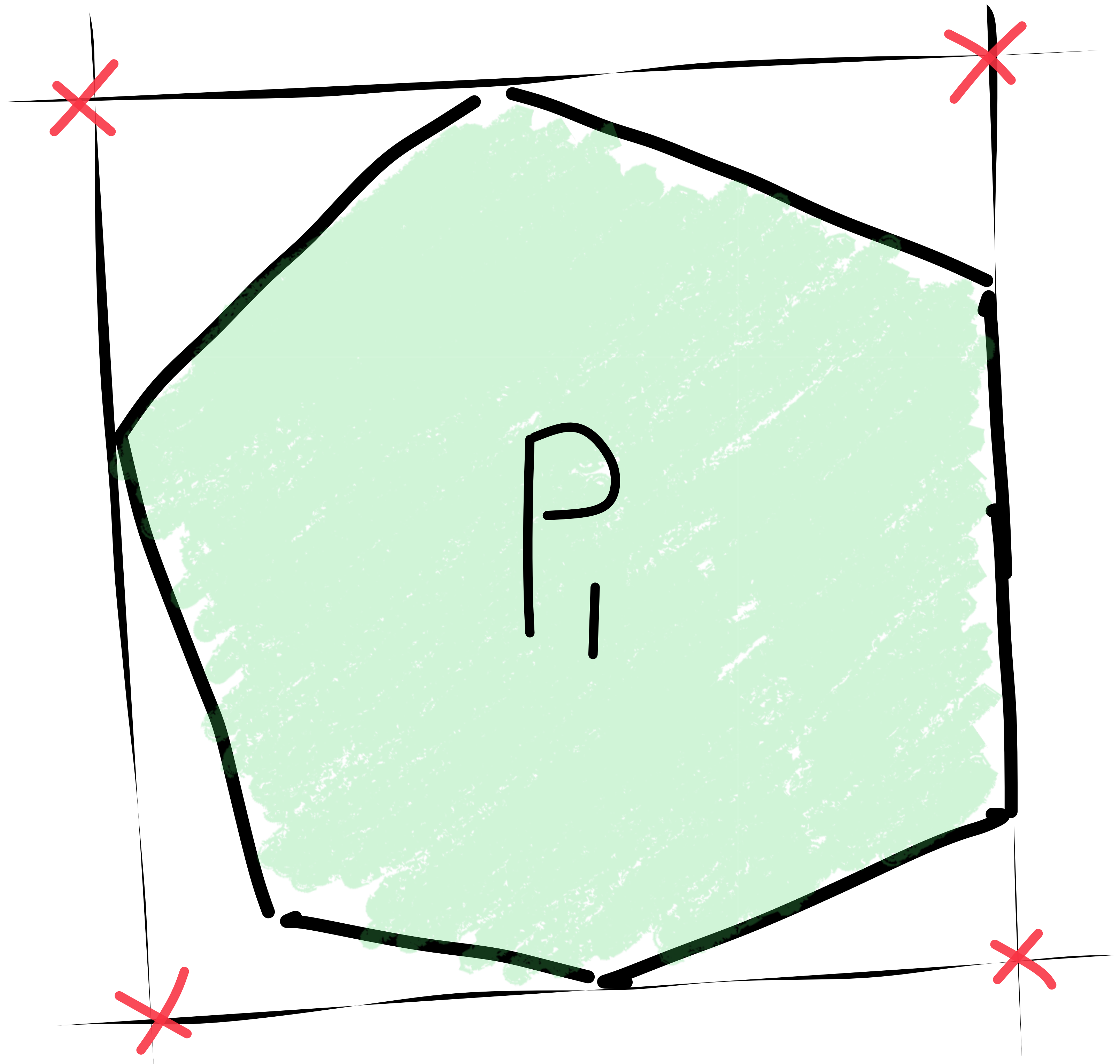
A CP proof that $P \cap \mathbb{Z}^n = \emptyset$ is a sequence of polytopes $P = P_1, \ldots, P_s = \emptyset$ s.t. P_{i+1} is deduced from P_i by a CG-cut



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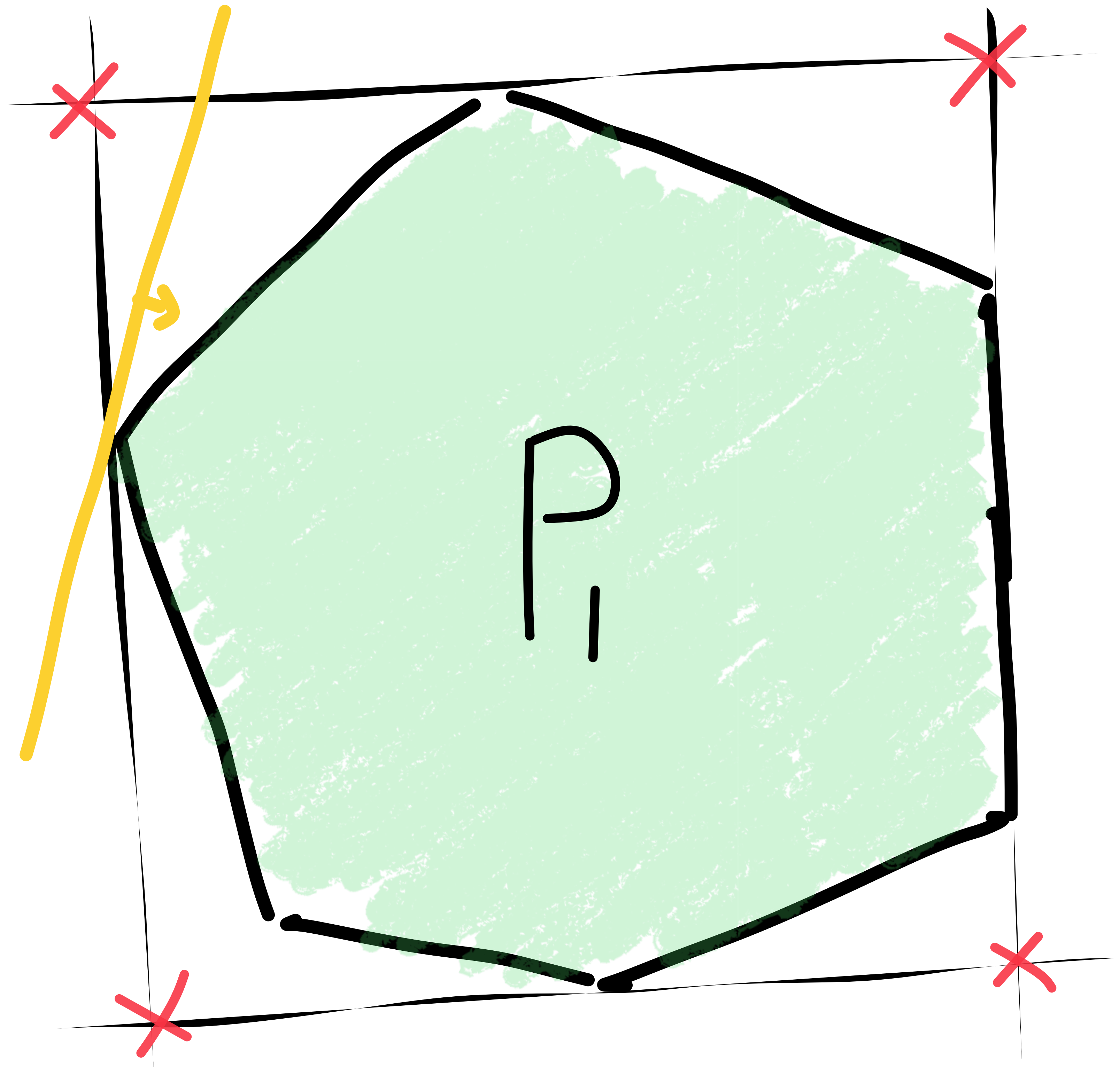

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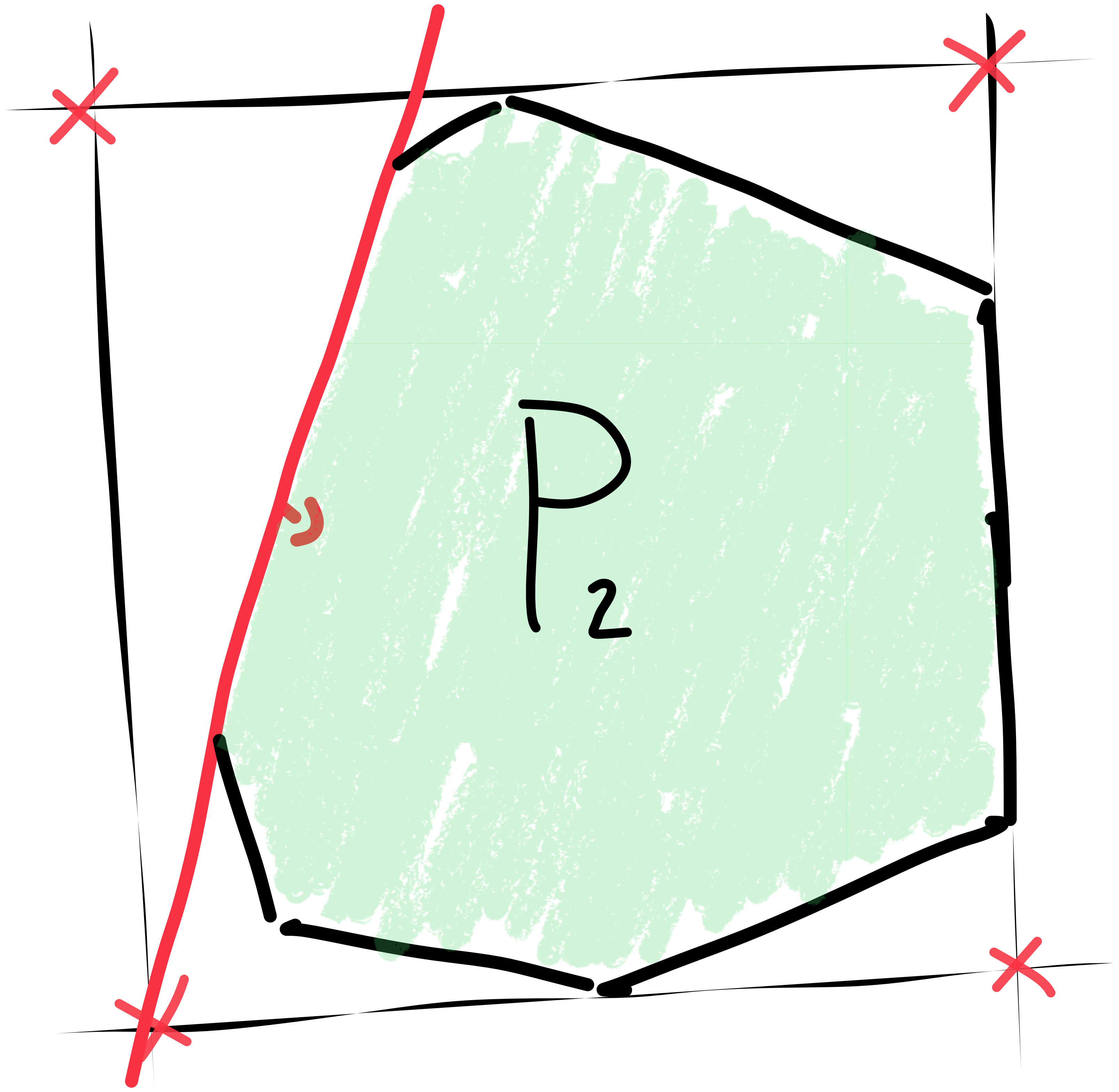

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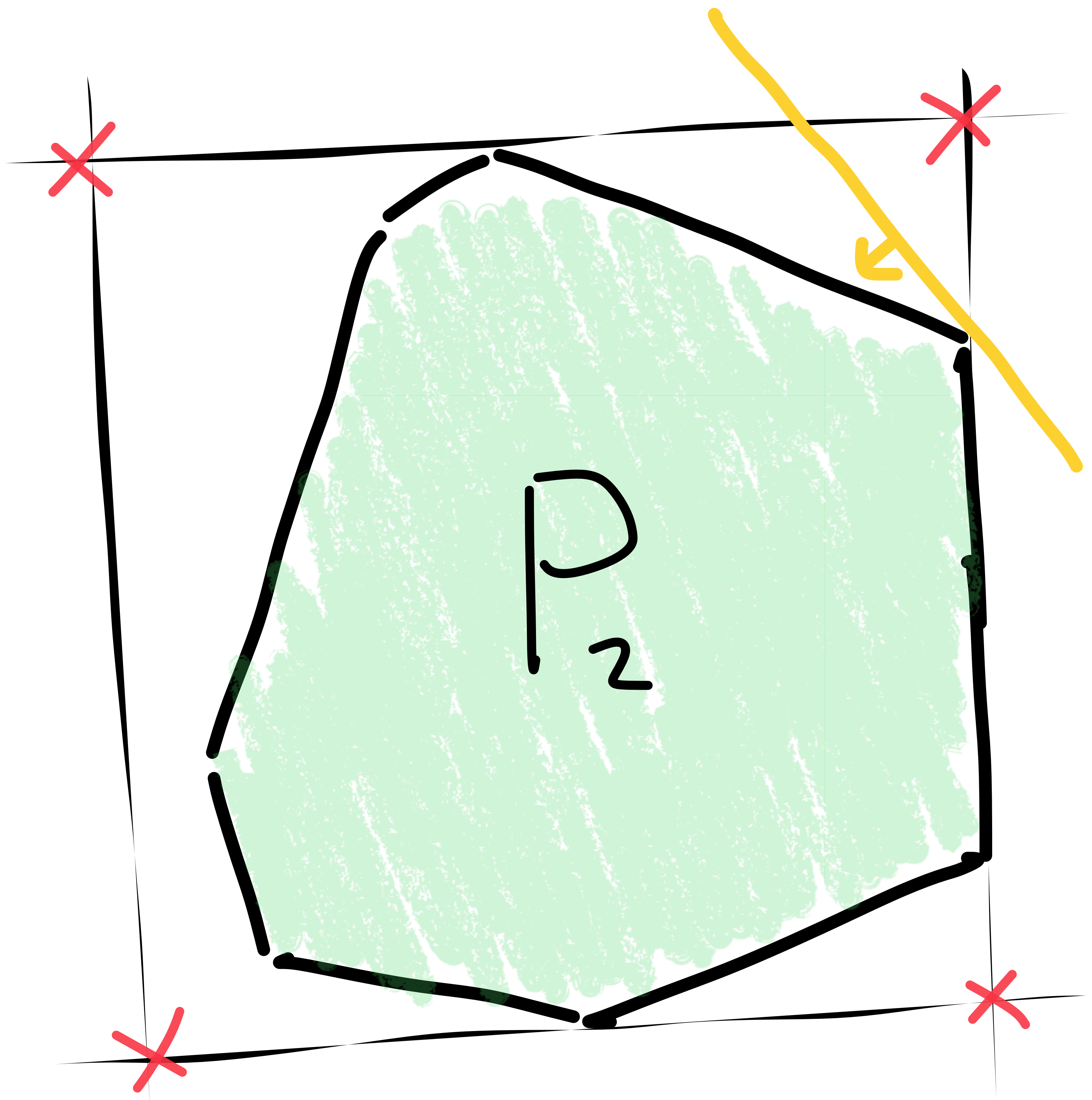

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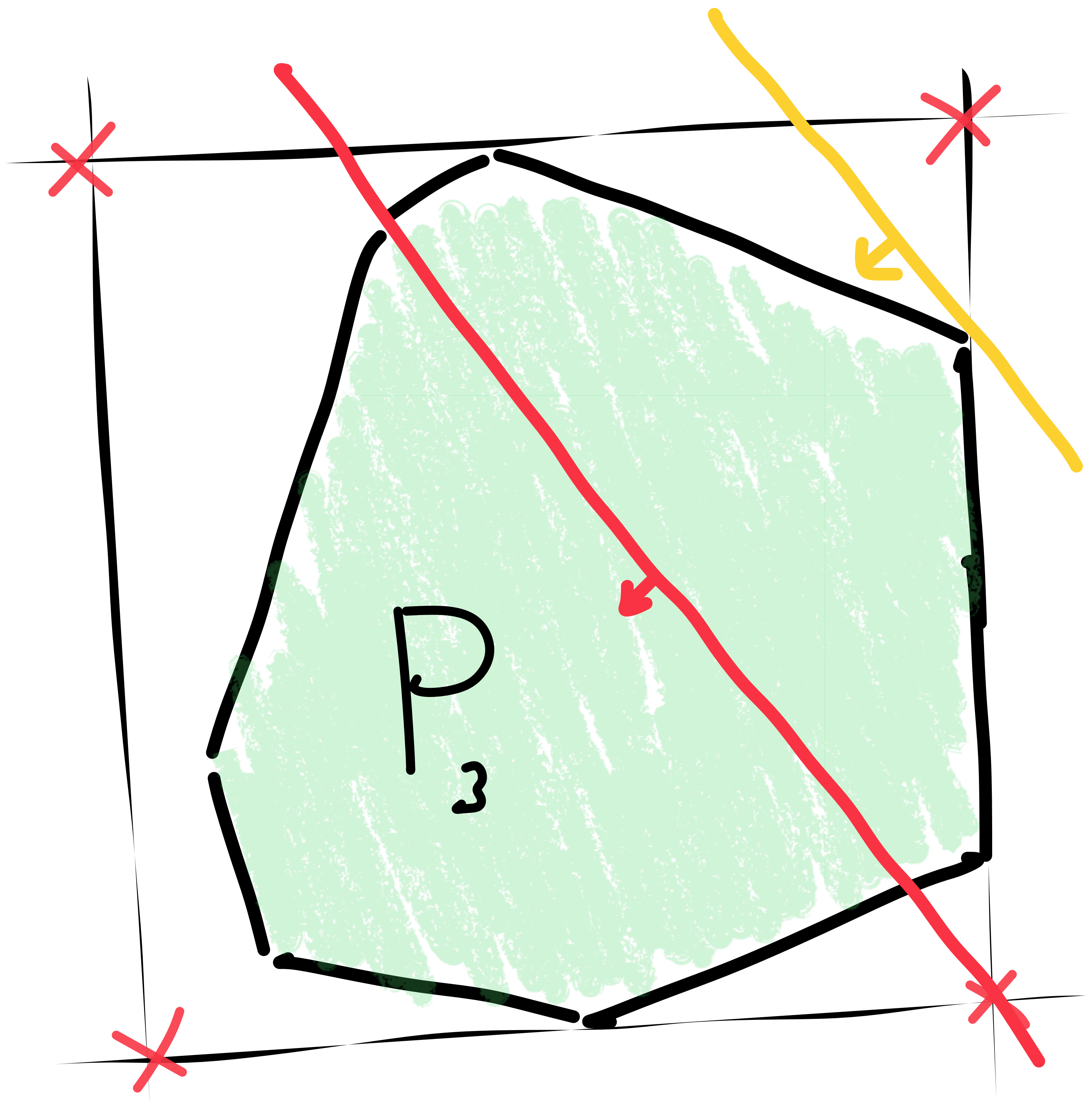

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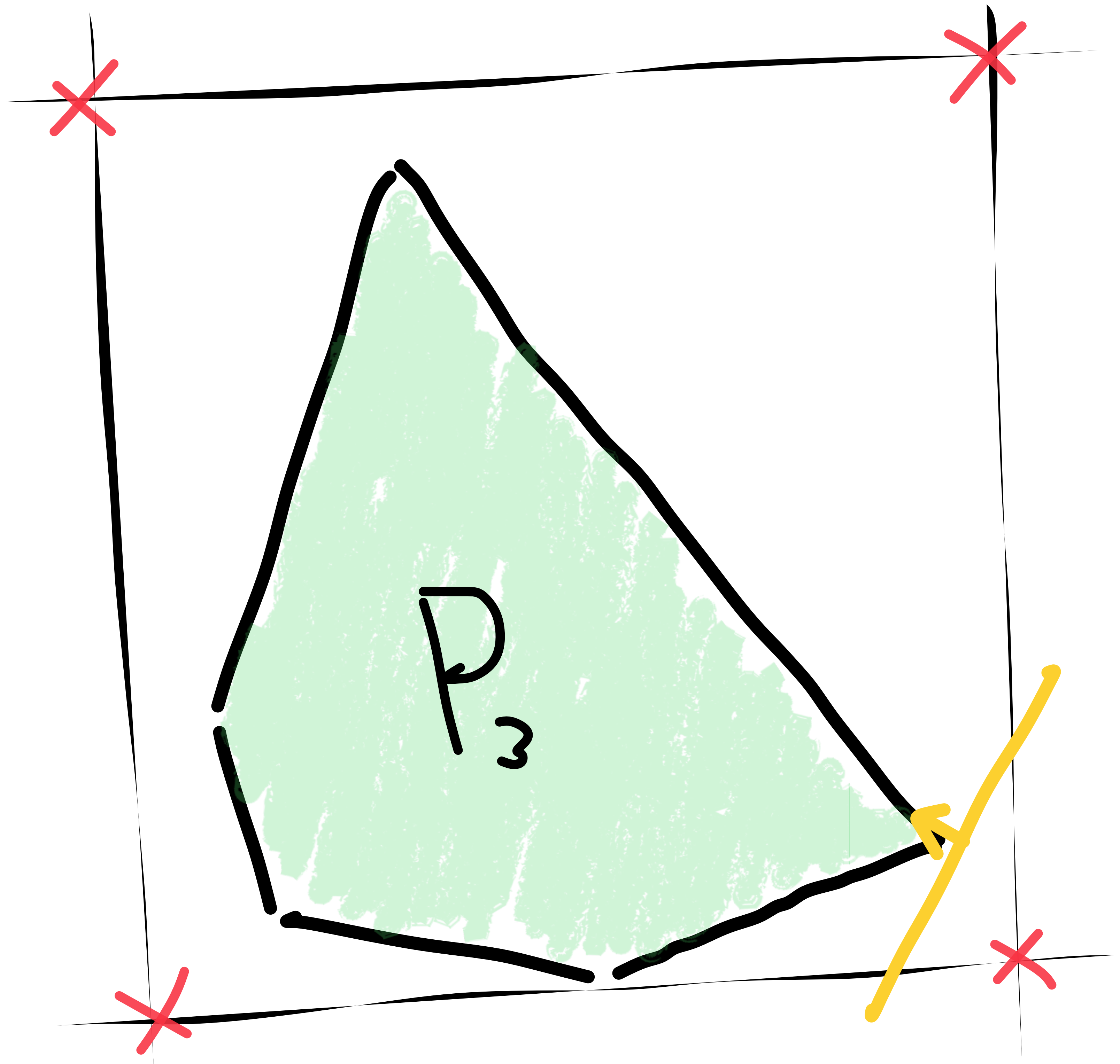

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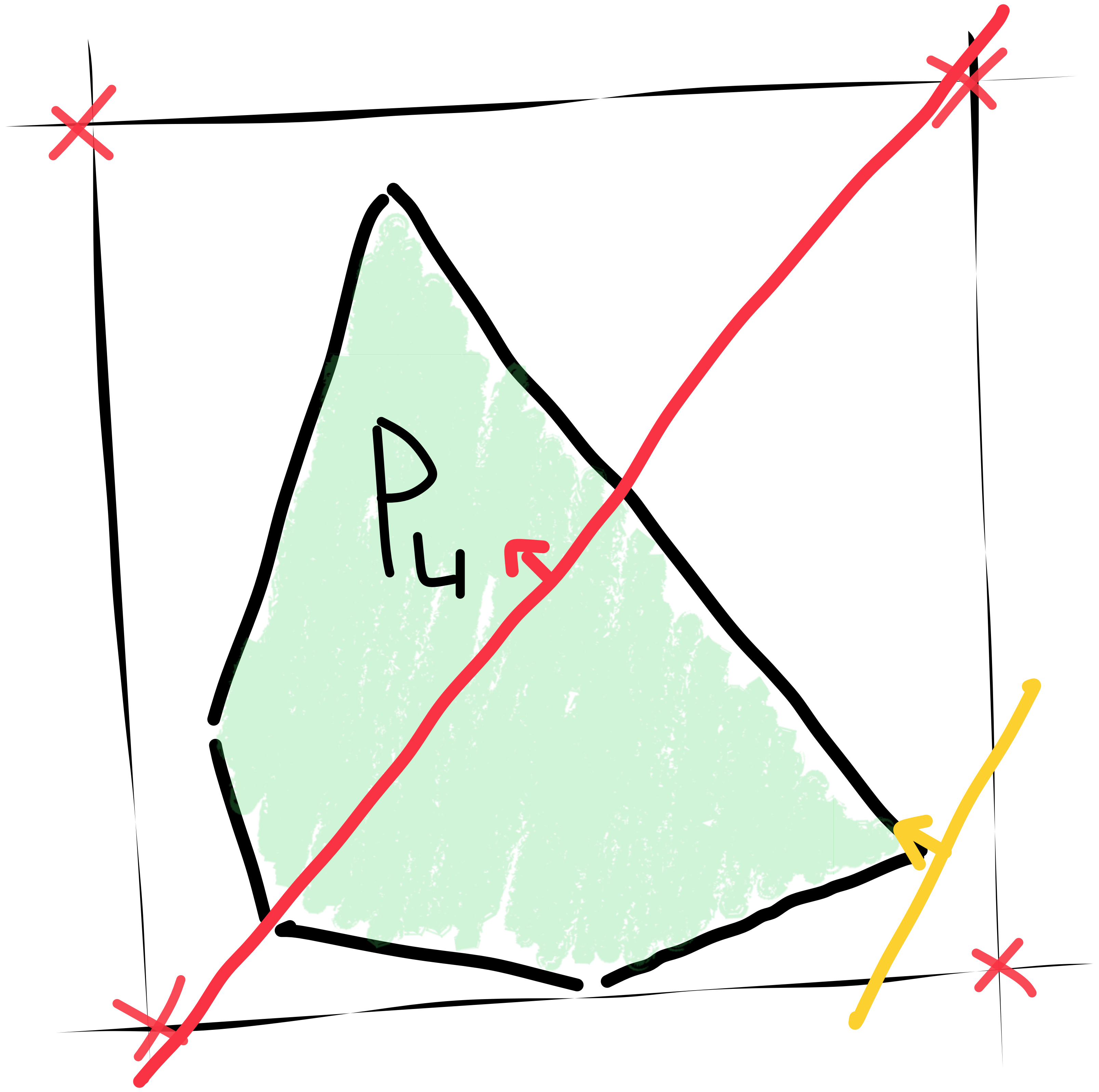

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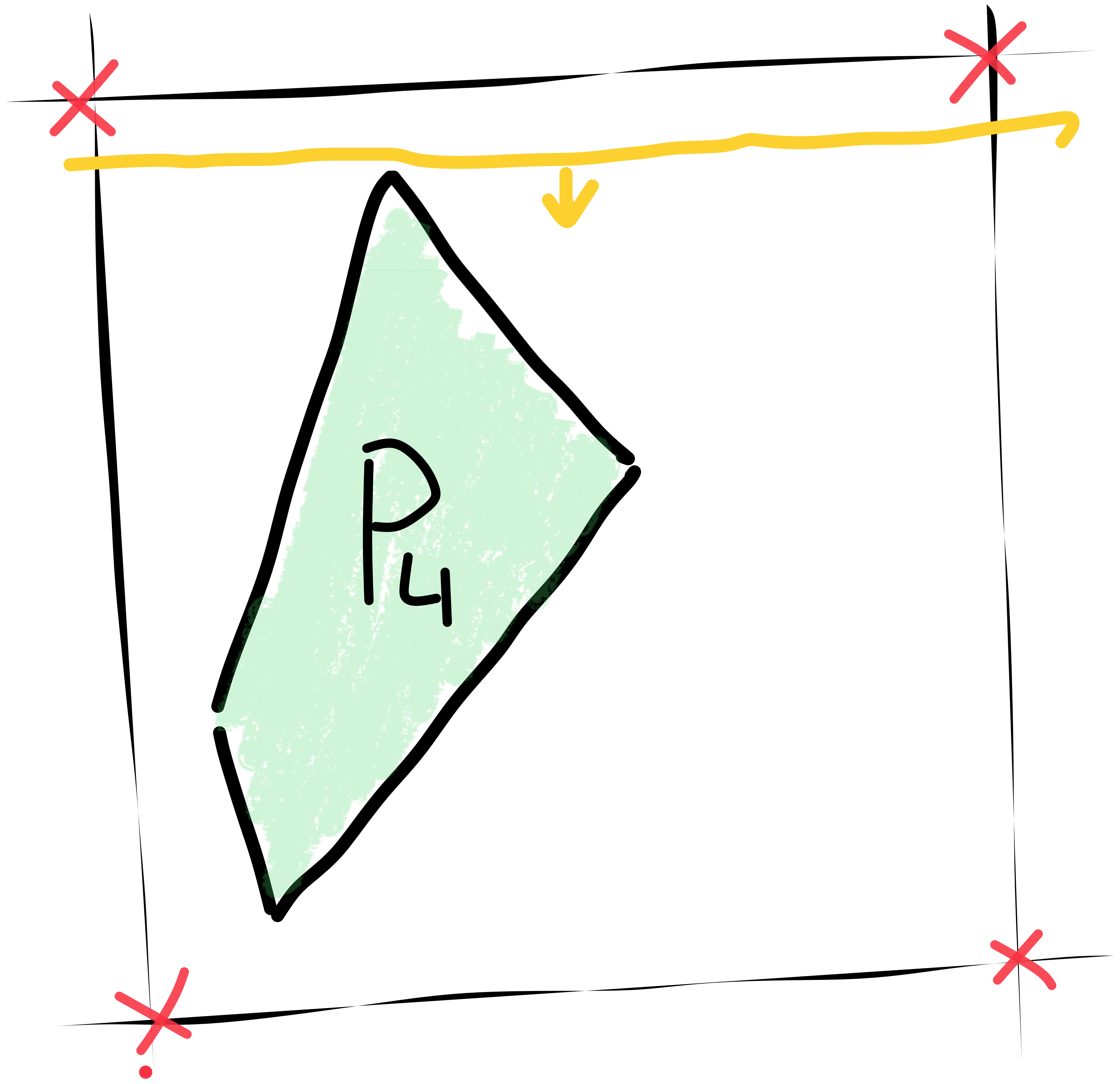

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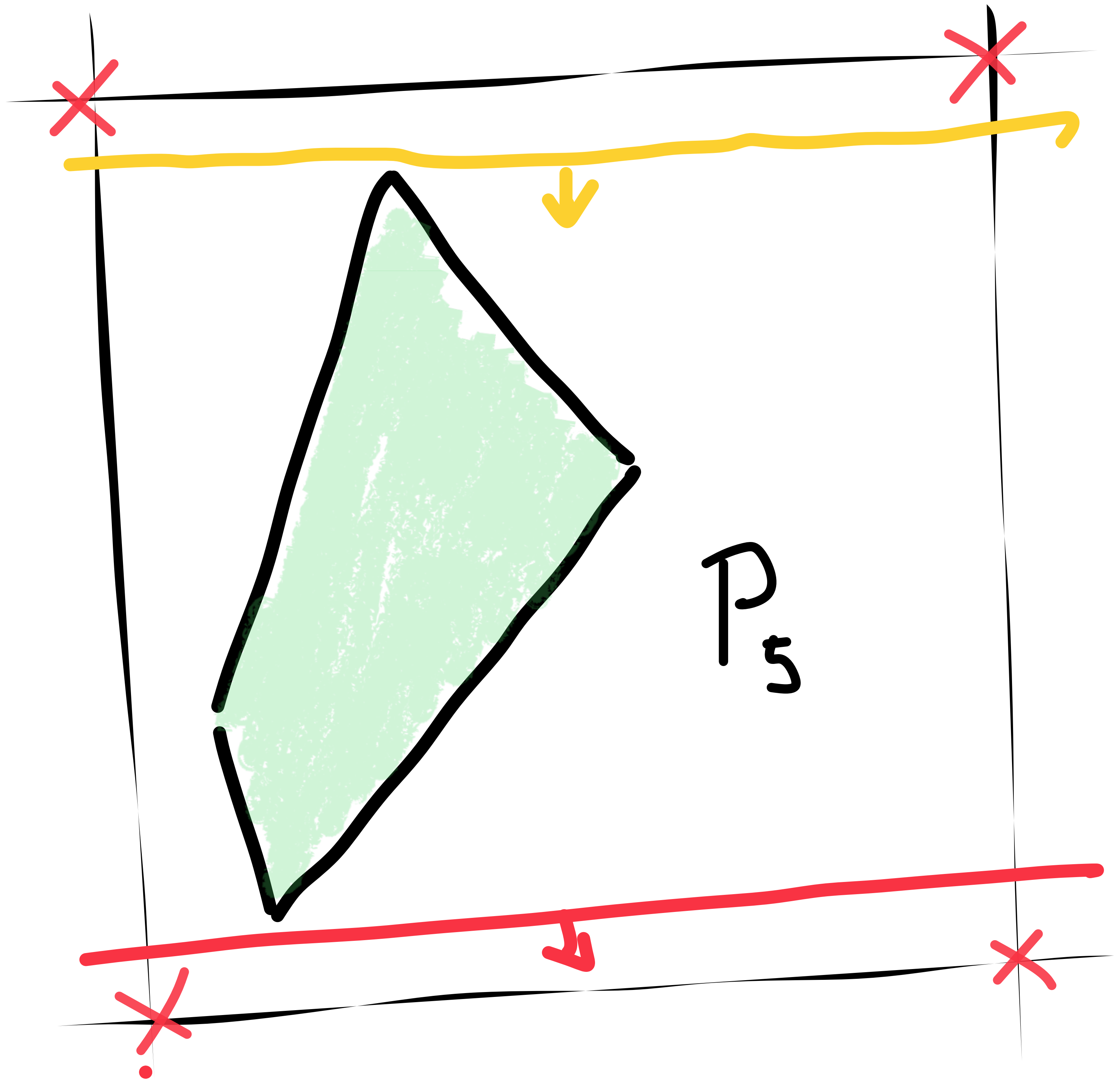

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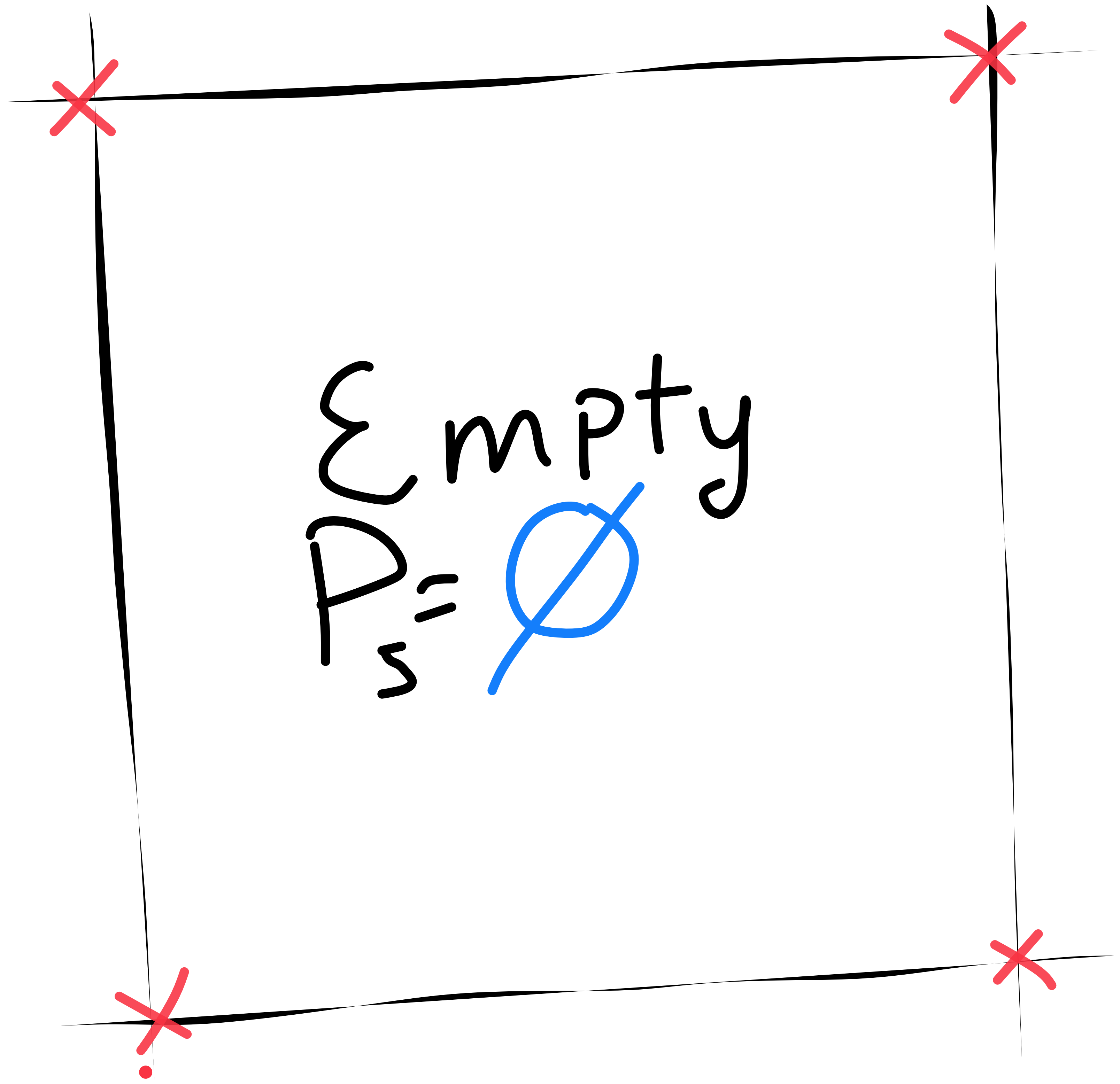

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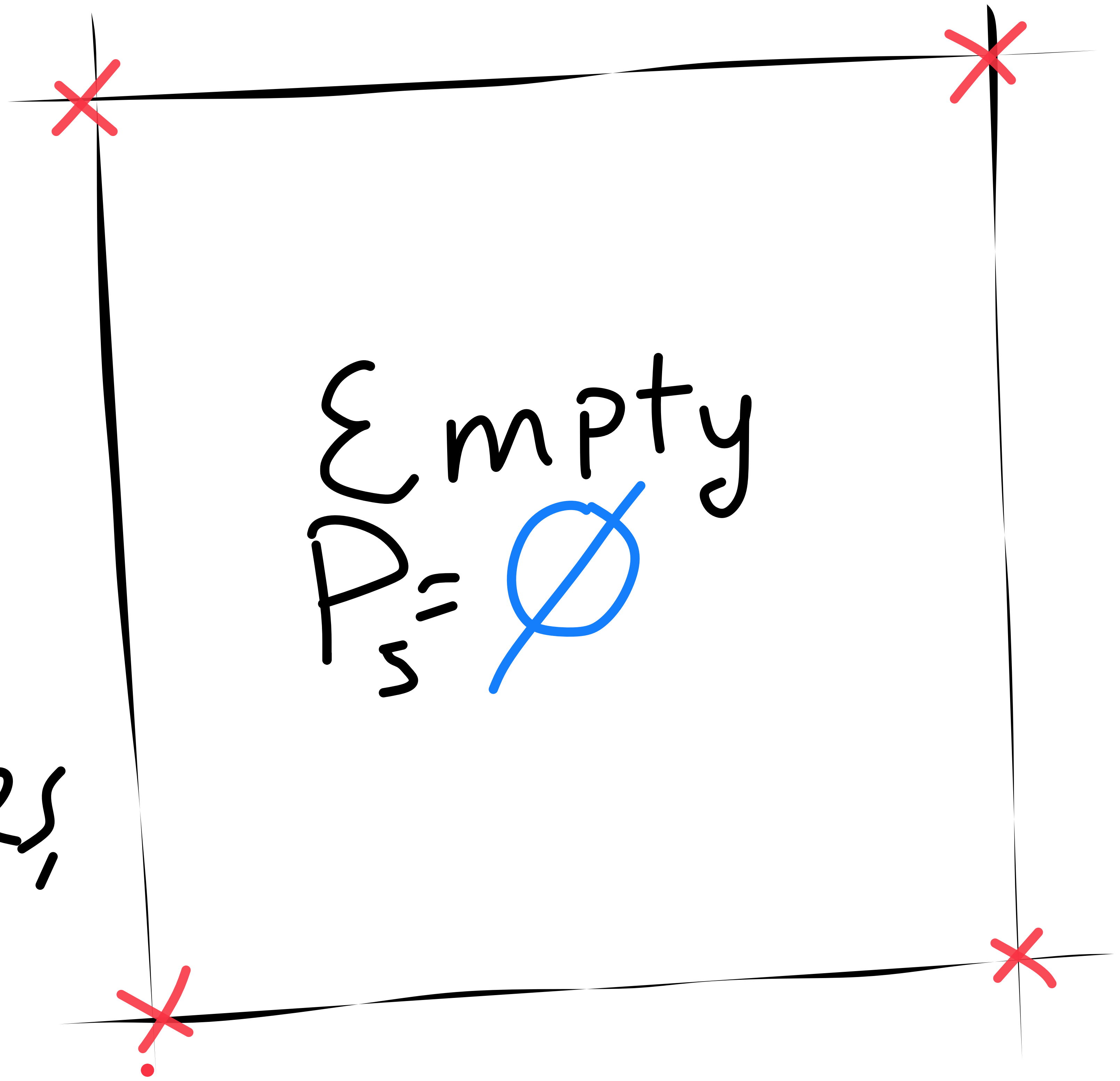
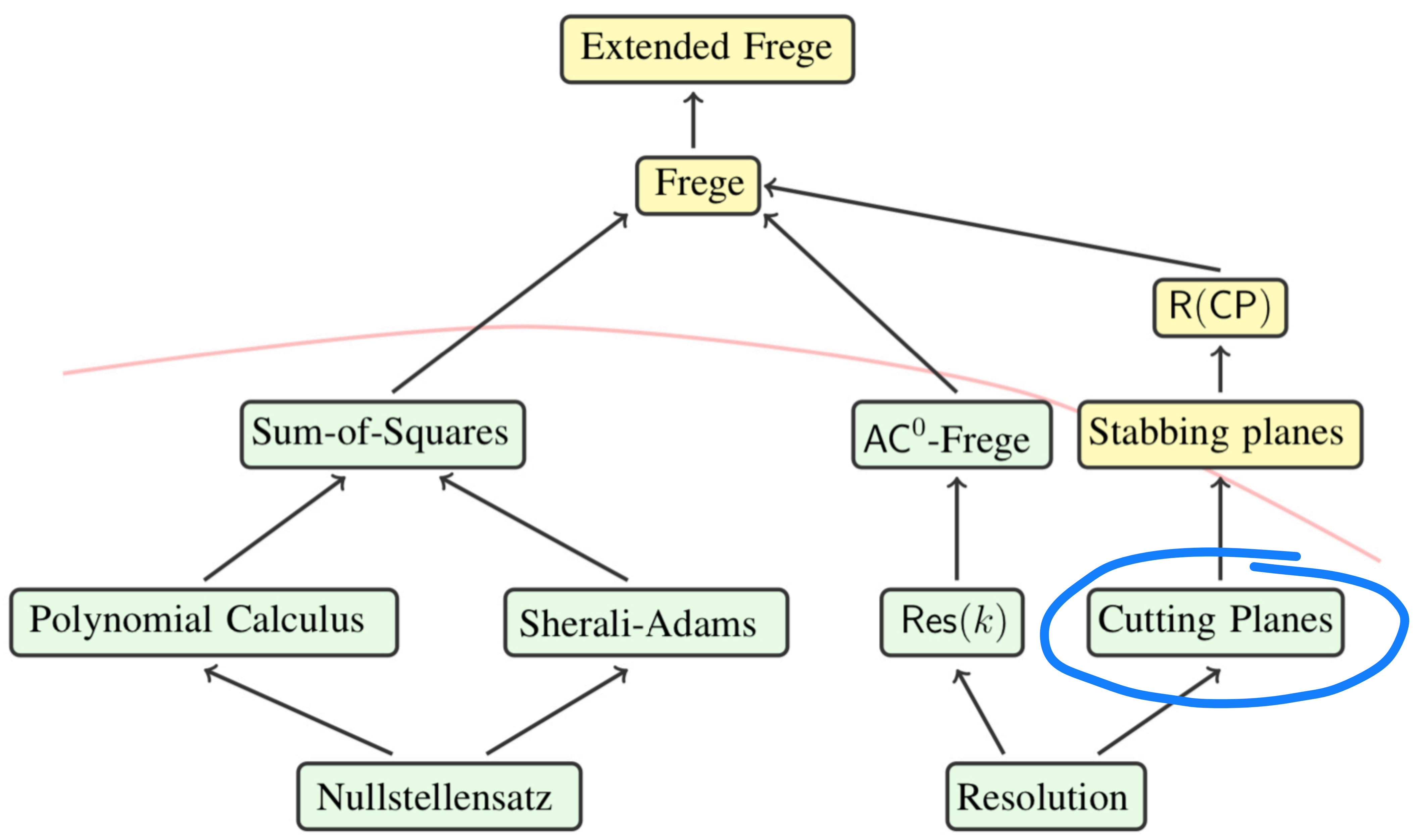


Proof Size: the number of polytopes, s

Cutting Planes

Powerful & Well-studied algebraic proof system



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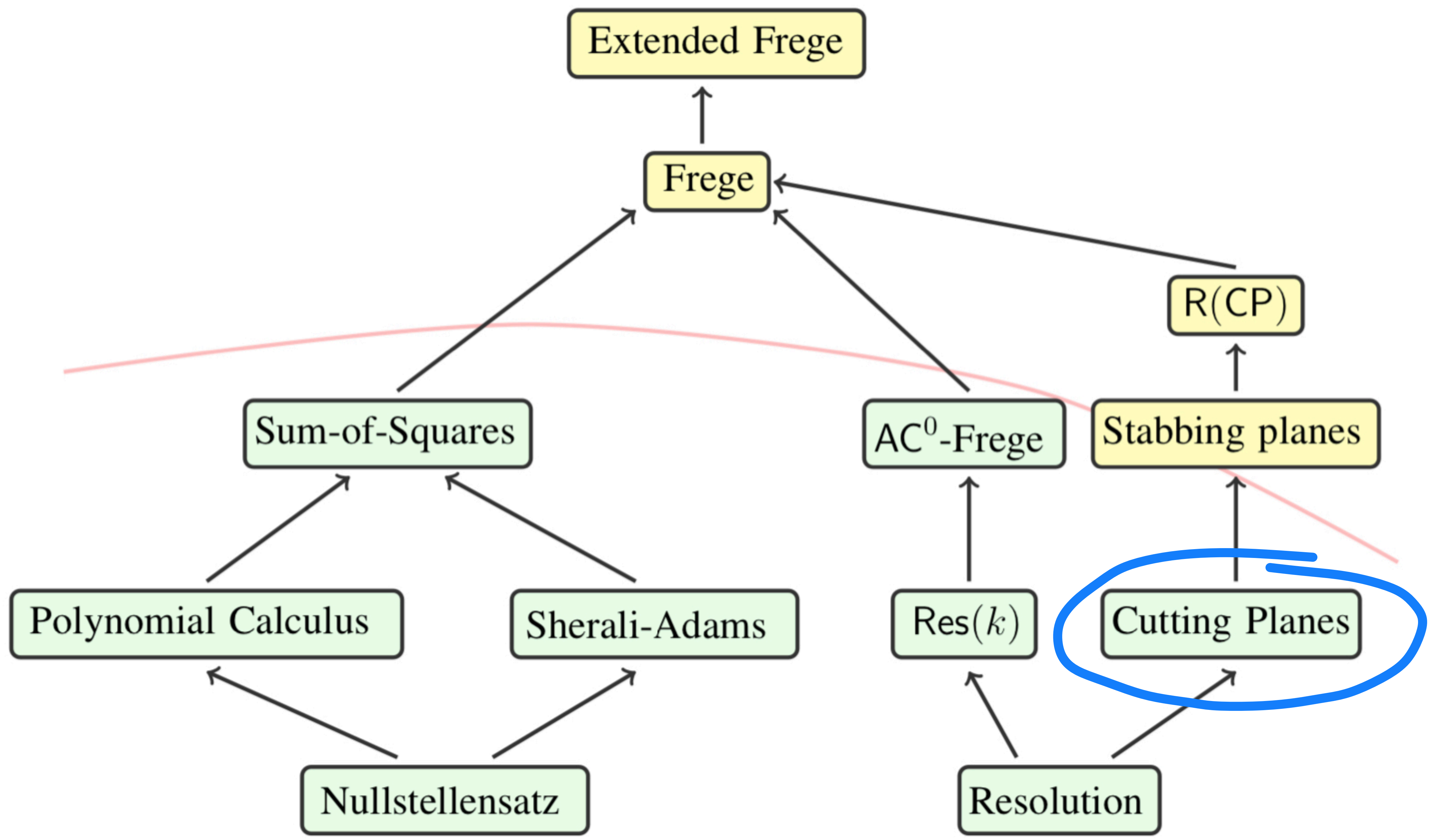

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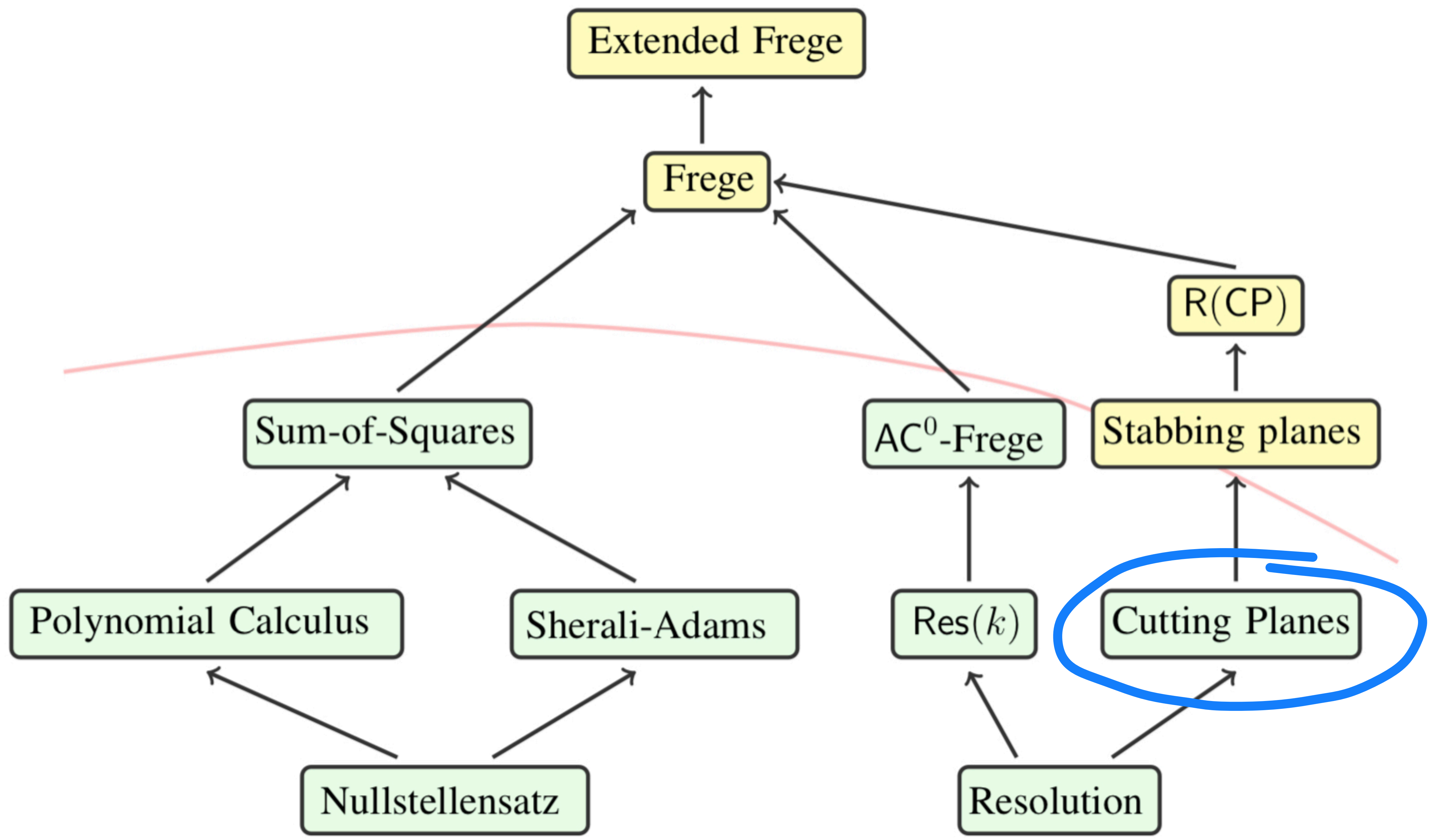
- ▷ Short proofs of PHP
- ▷ Exponential lowerbounds [Pud97]



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Modern IP Algorithms combine cutting planes with a branch-and-bound procedure

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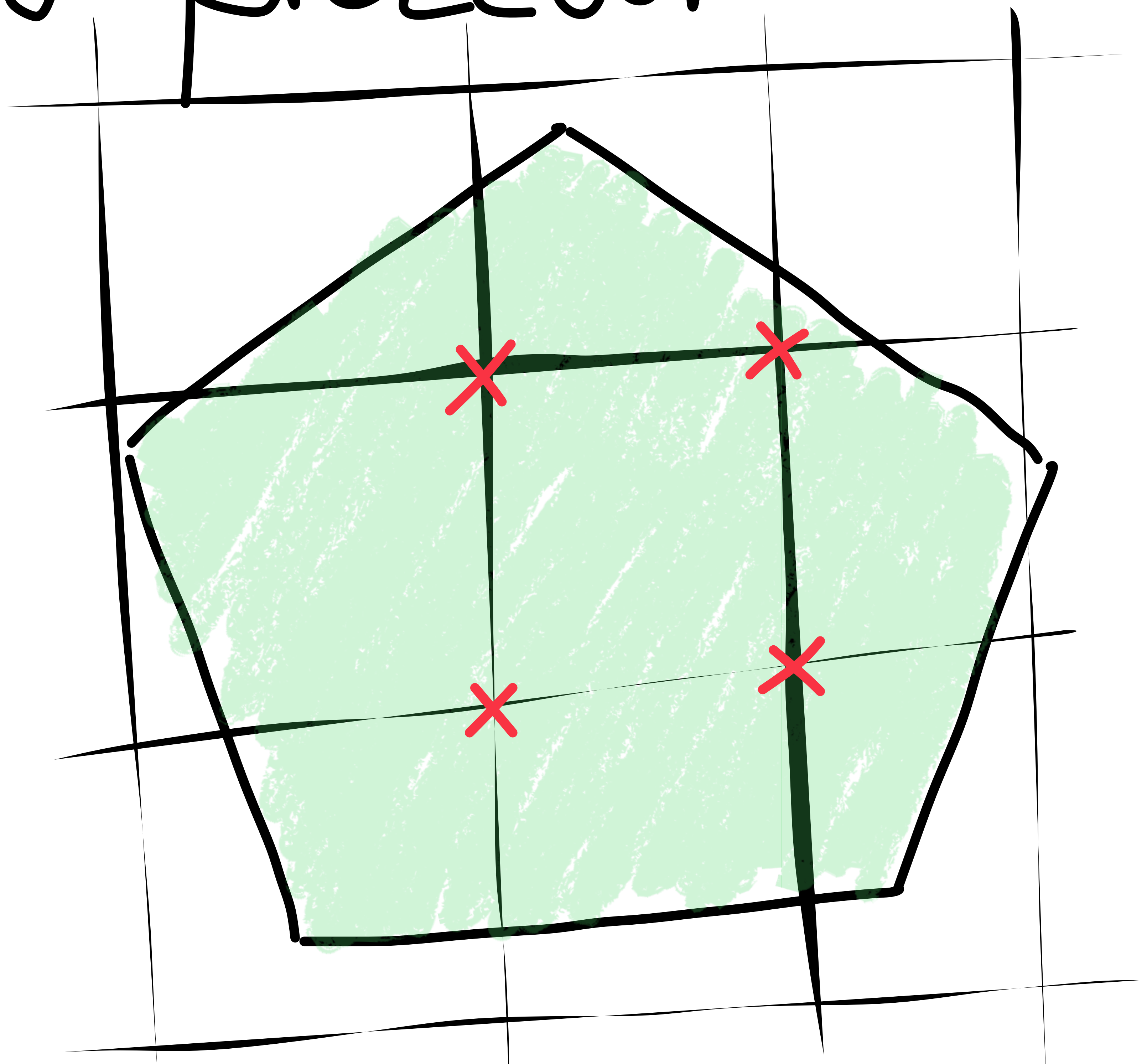

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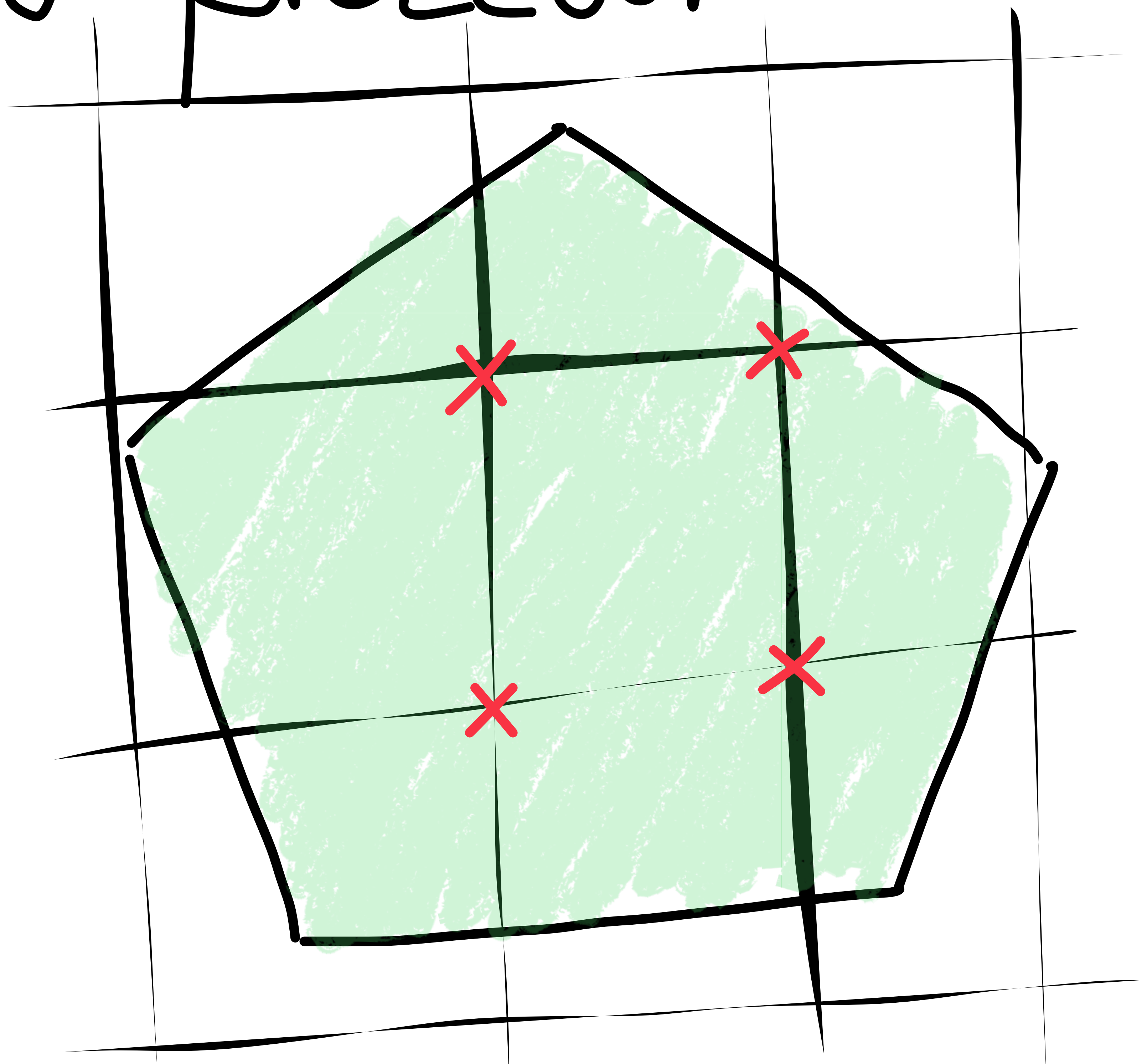

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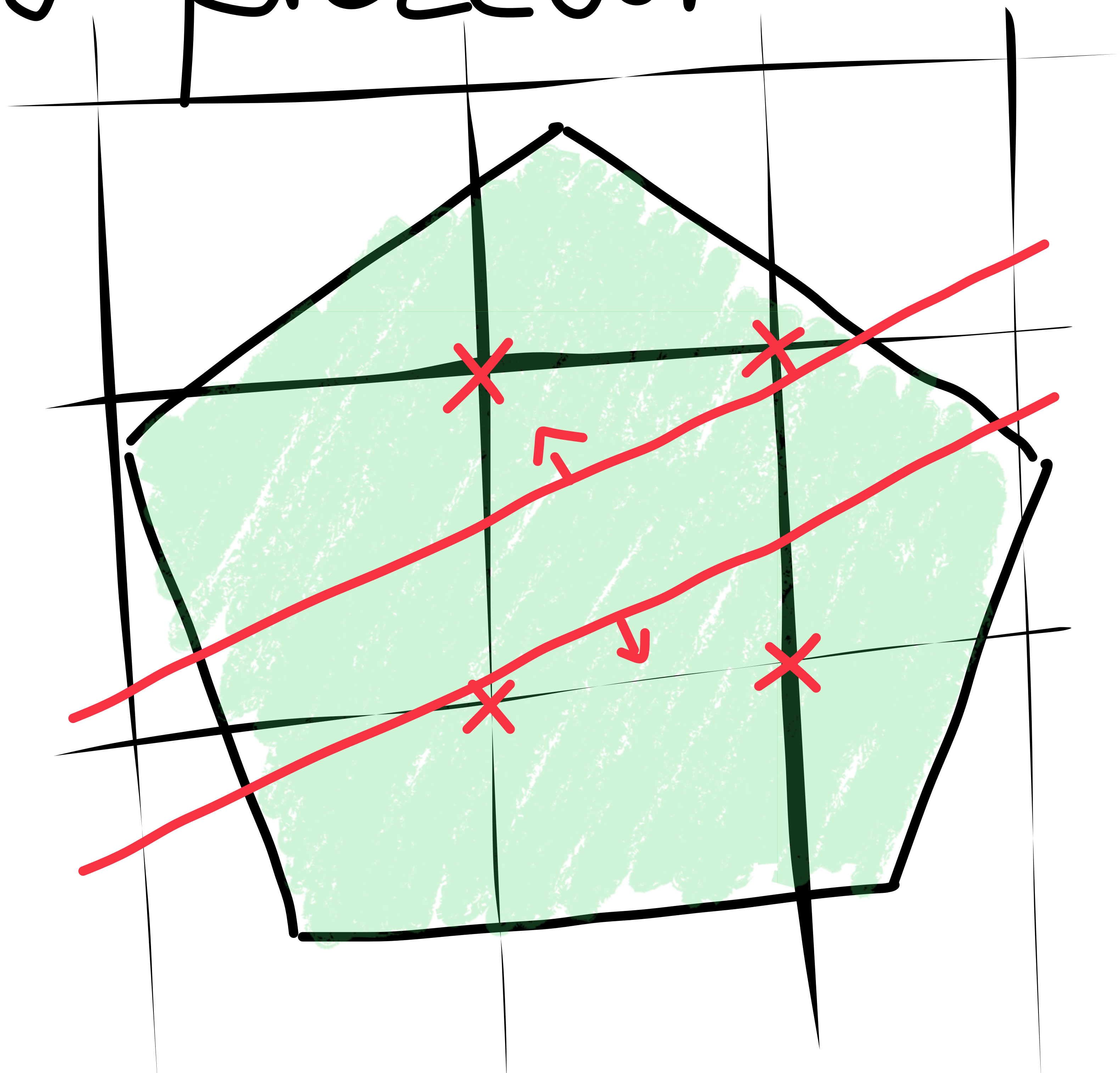

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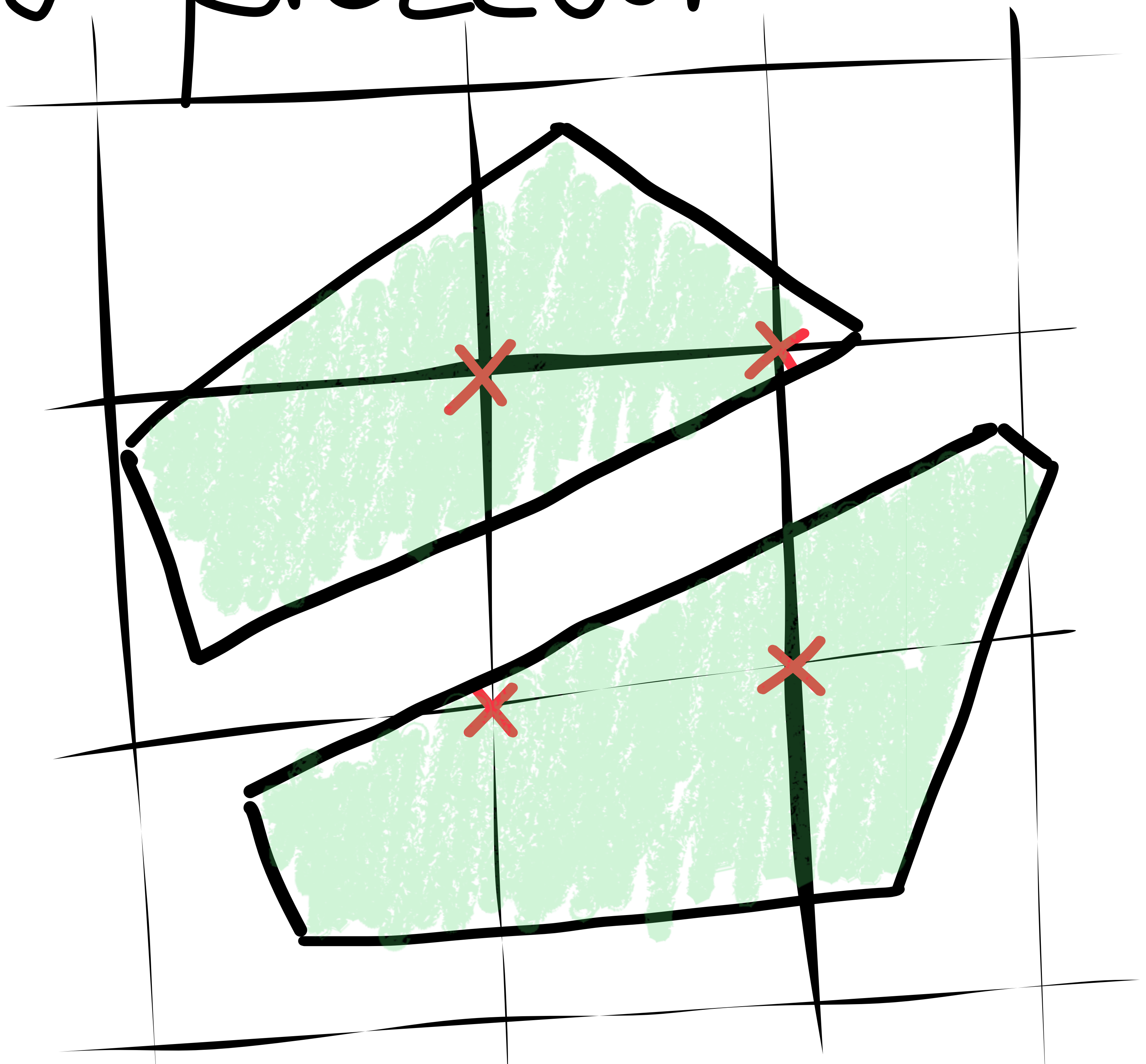

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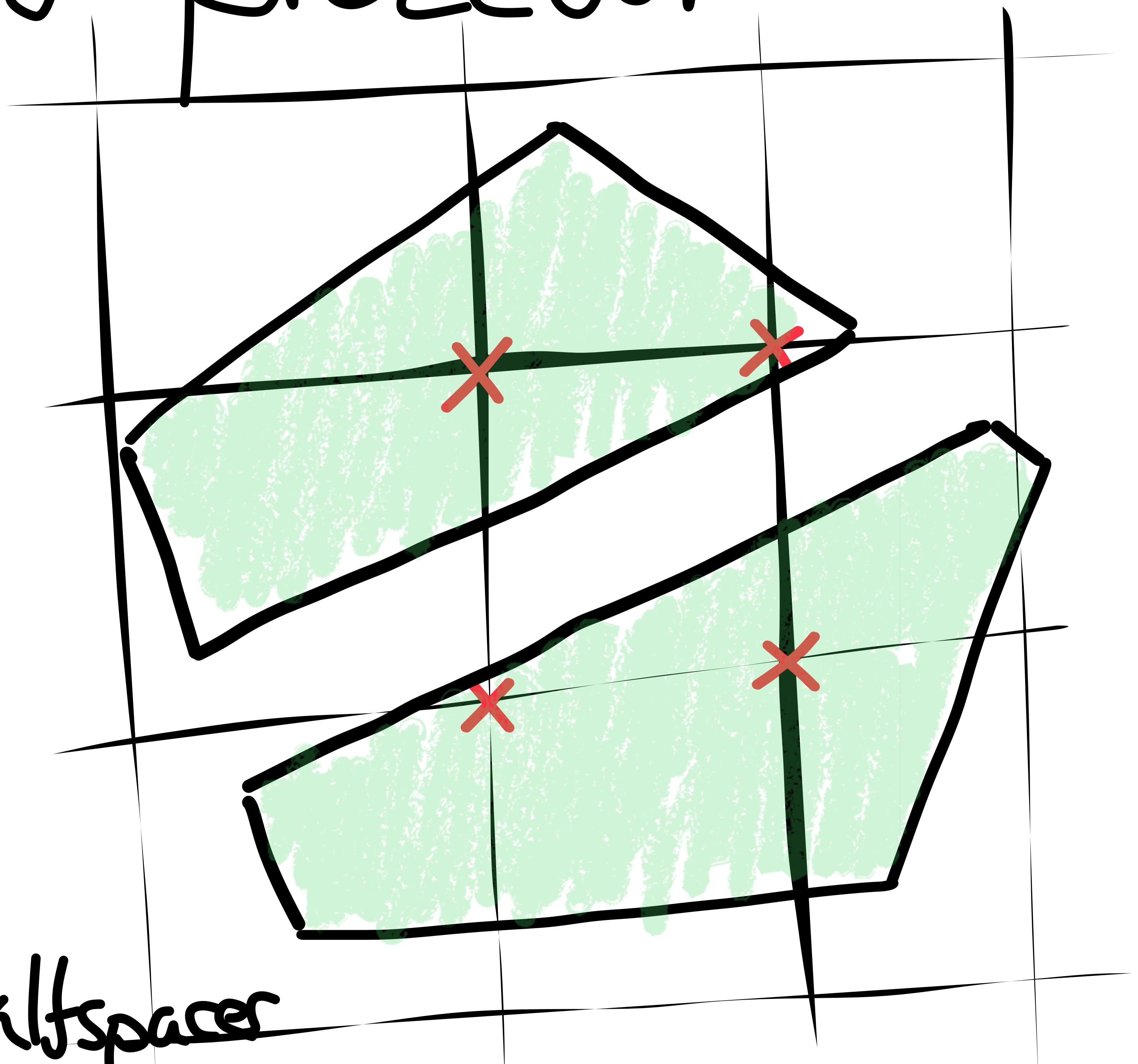
▷ In practice, P is broken into $P \cap \{x : ax \geq b\}$ and $P \cap \{x : ax \leq b-1\}$ for some class of halfspaces



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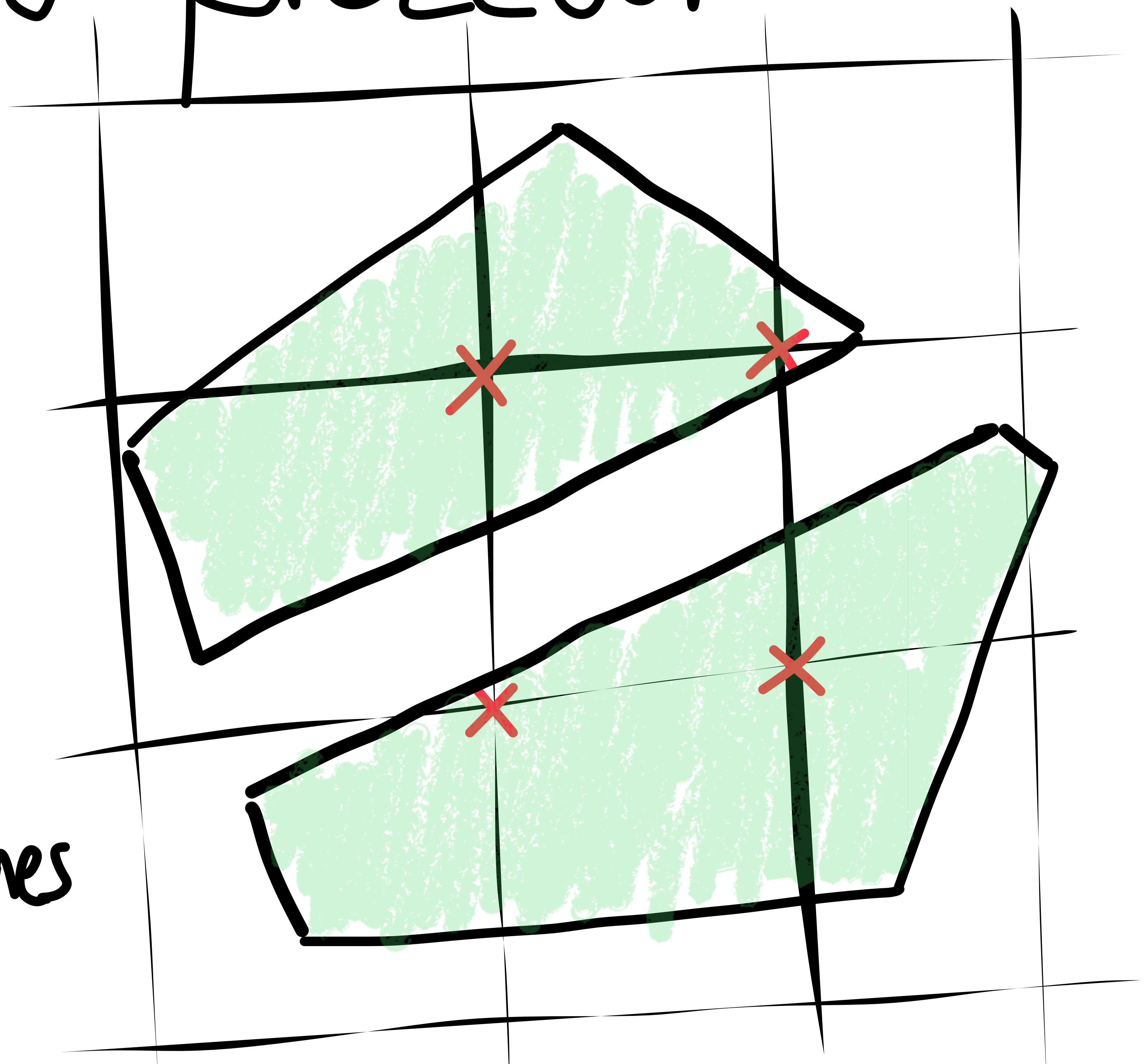

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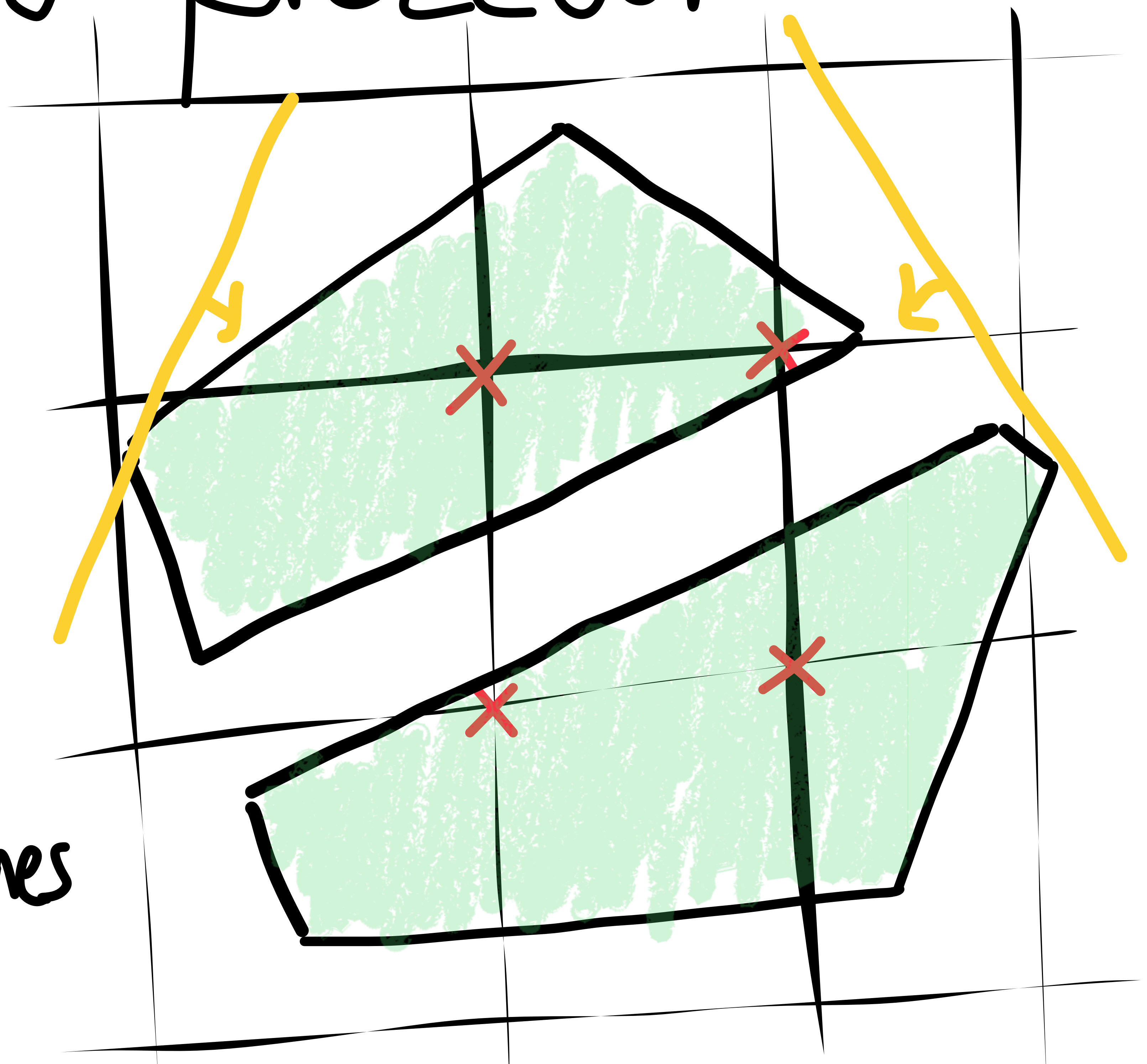

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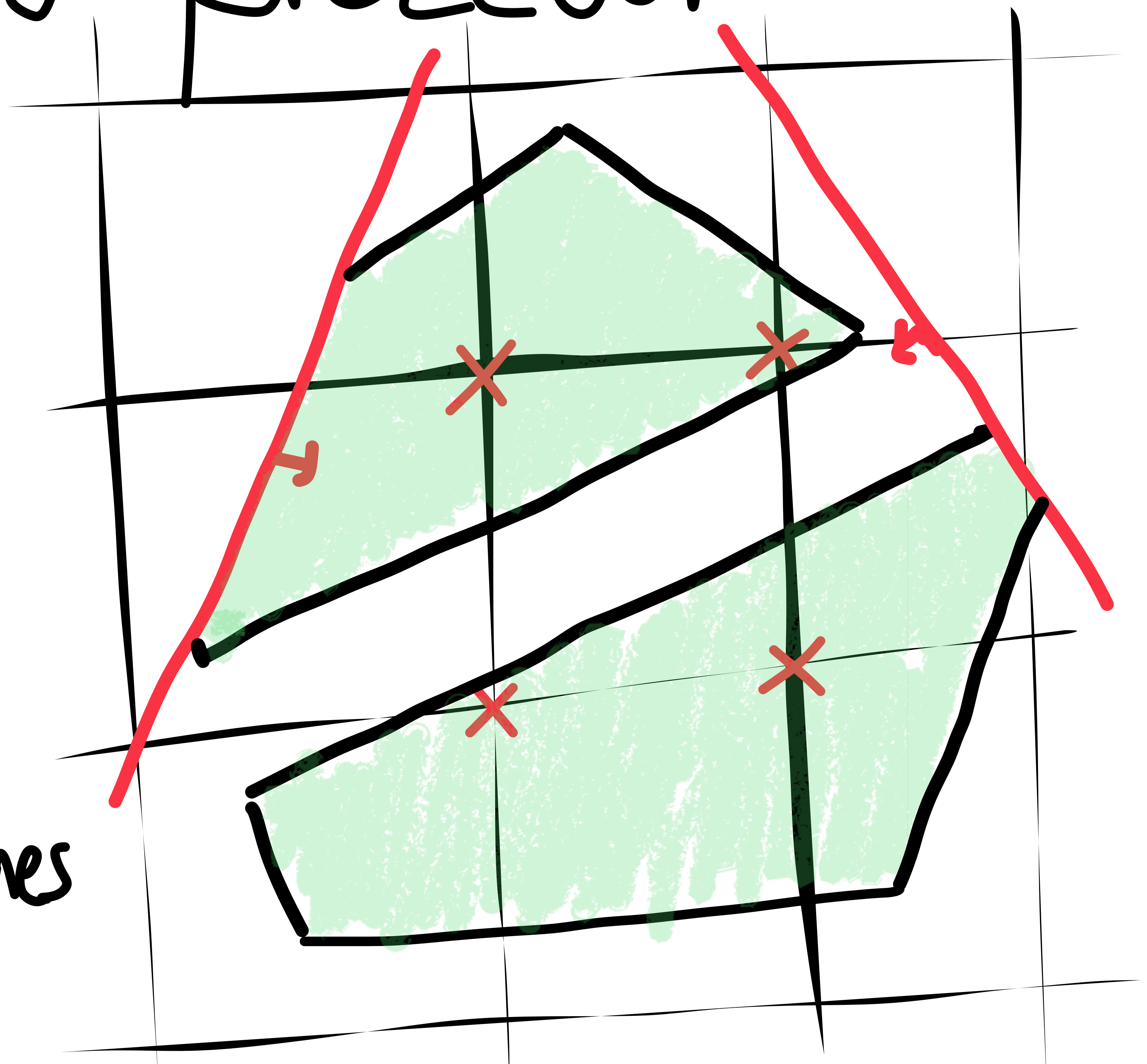

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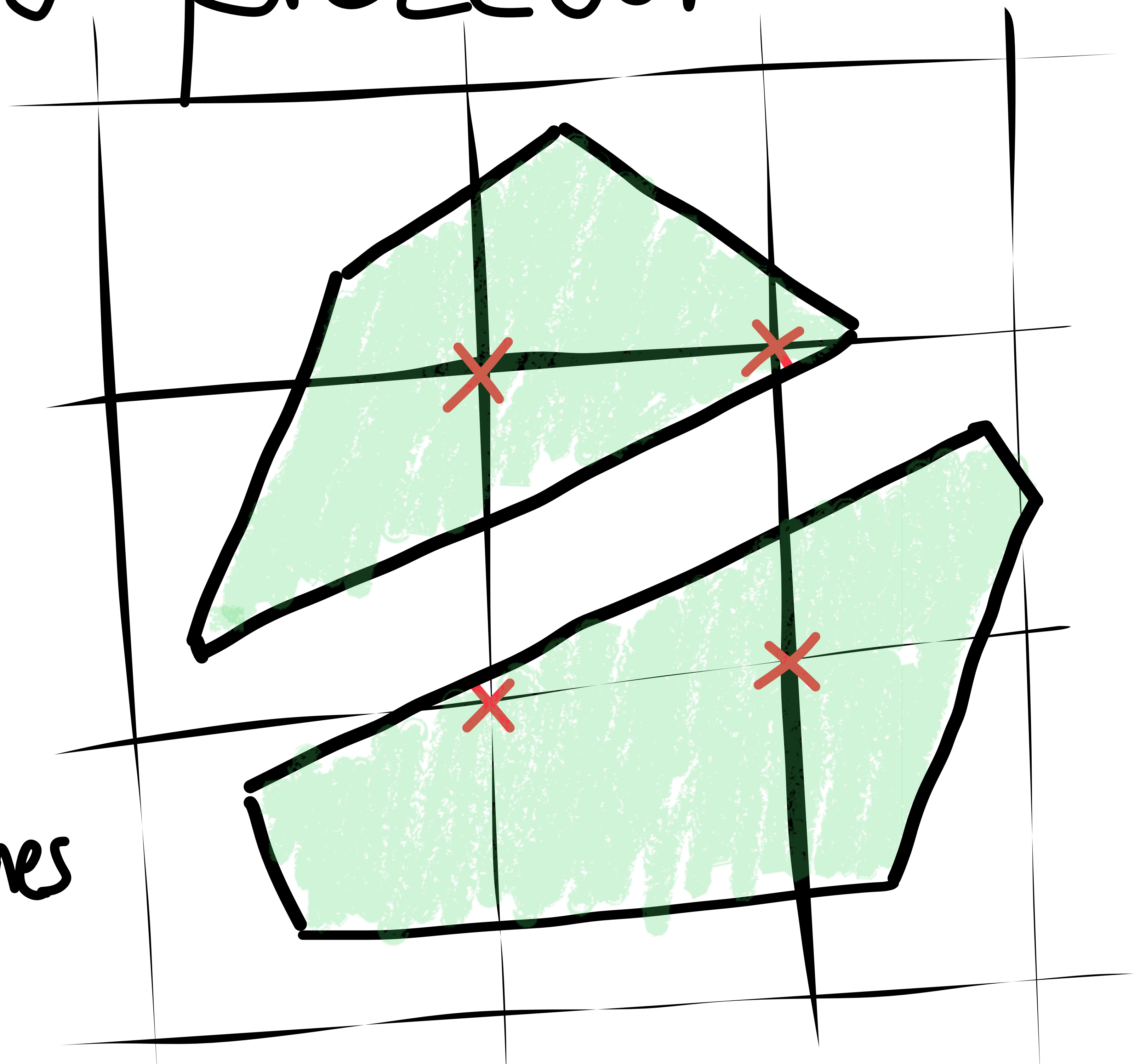

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Repeat

Stabbing Planes [BFI+18]

- Formalizes practical branch-and-cut as a proof system

Stabbing Planes [BFI+18]

- Formalizes practical branch-and-cut as a proof system
- Extends DPLL to reason about linear inequalities

DPLL Refutation

$$\{x_1 \vee x_2,\ \bar{x}_1 \vee x_2,\ x_1 \vee \bar{x}_2,\ \bar{x}_1 \vee \bar{x}_2\}$$

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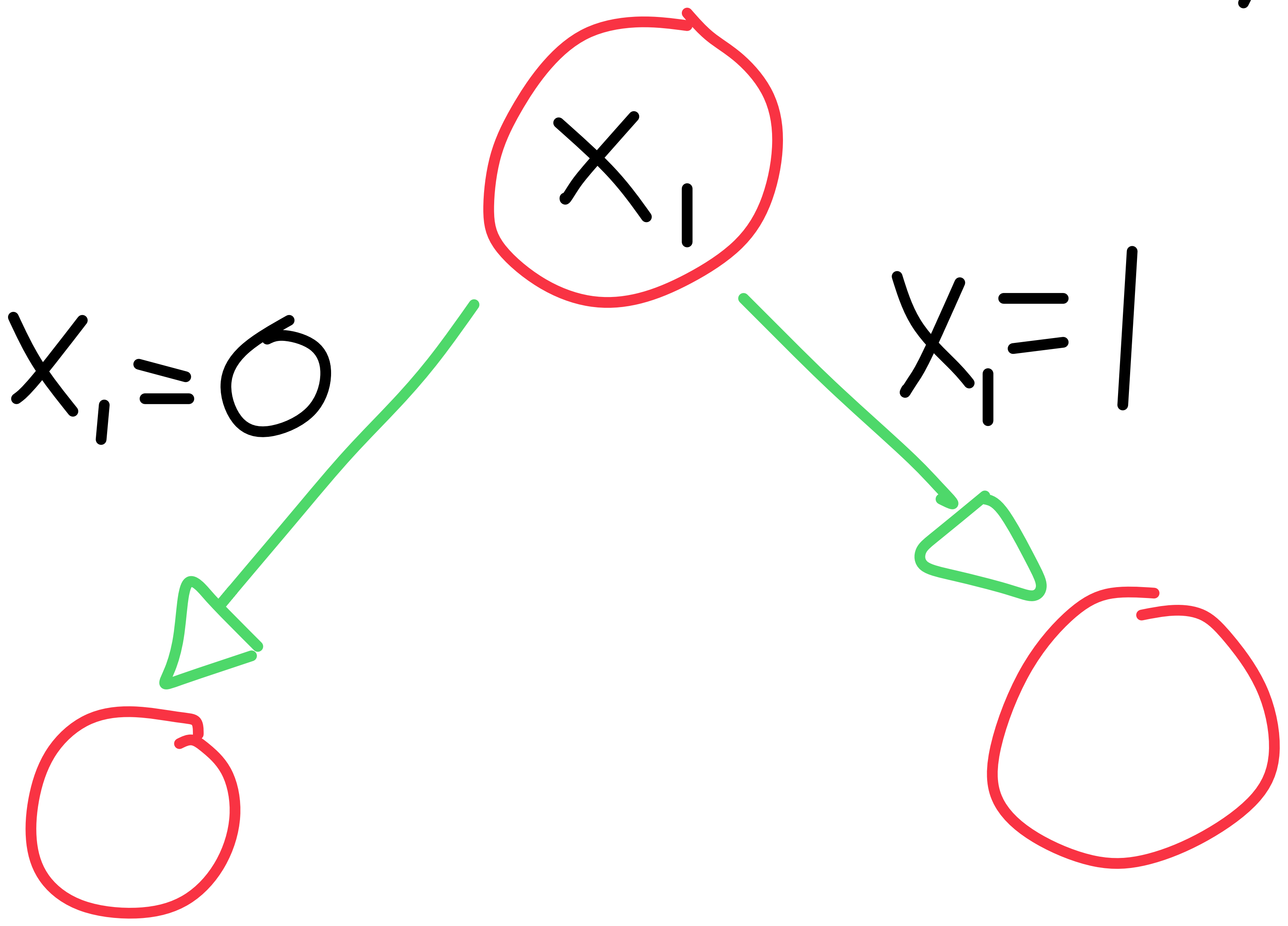

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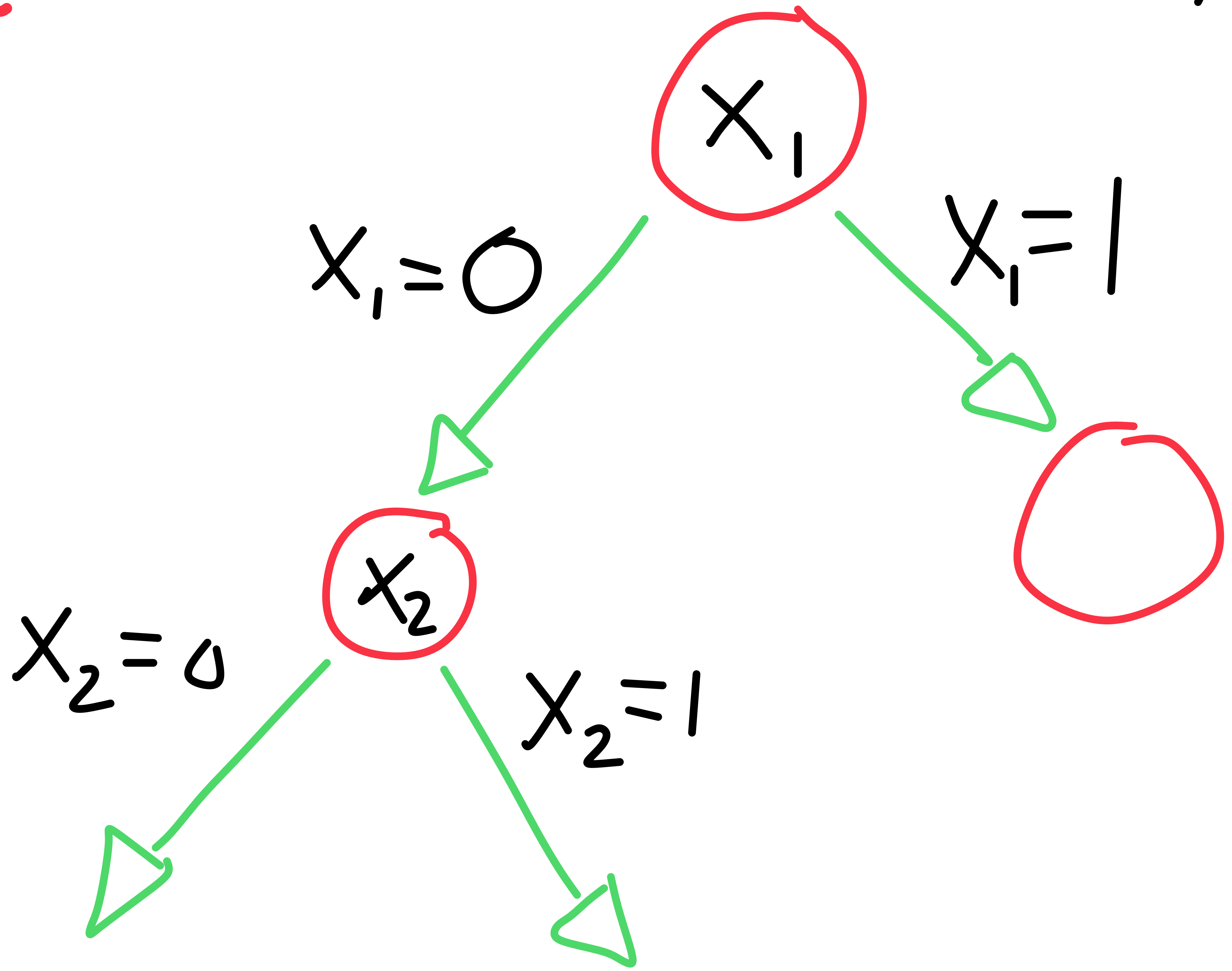

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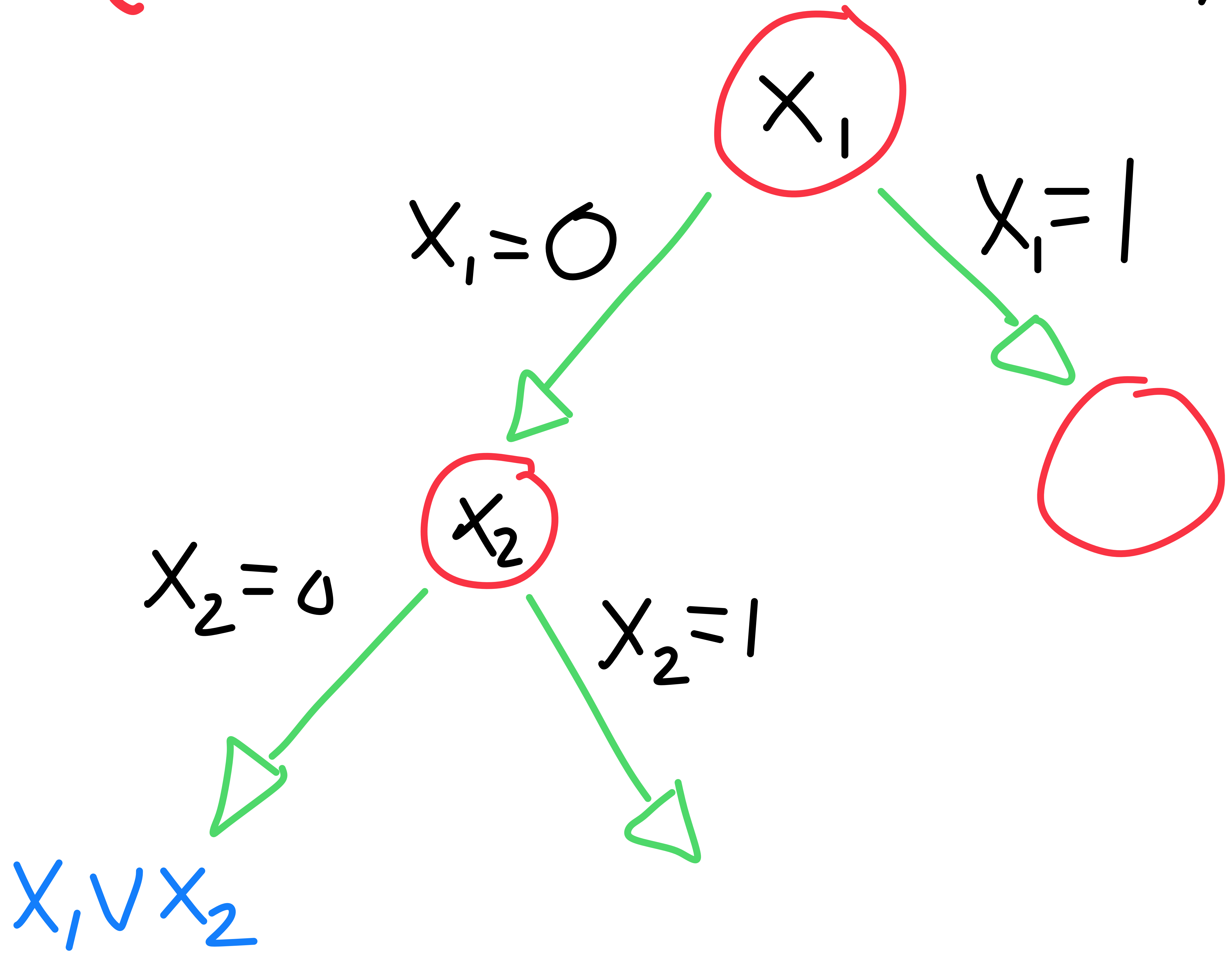

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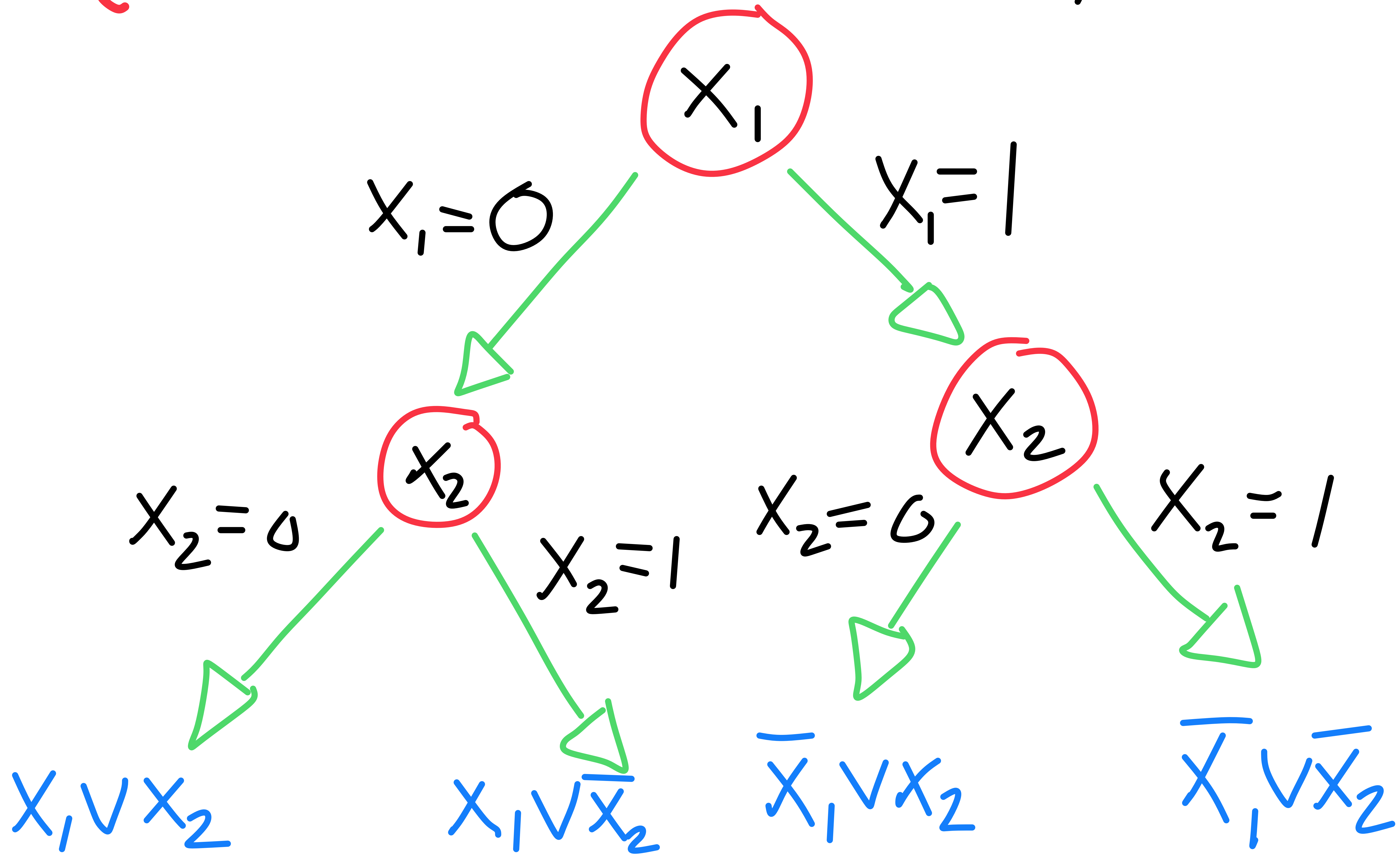

DPLL as Polytopes

$P = \{x_1 + x_2 \geq 1,\ x_1 - x_2 \geq 0,\ x_2 - x_1 \geq 0,\ -x_1 - x_2 \geq -1,\ 0 \leq x_i \leq 1\}$

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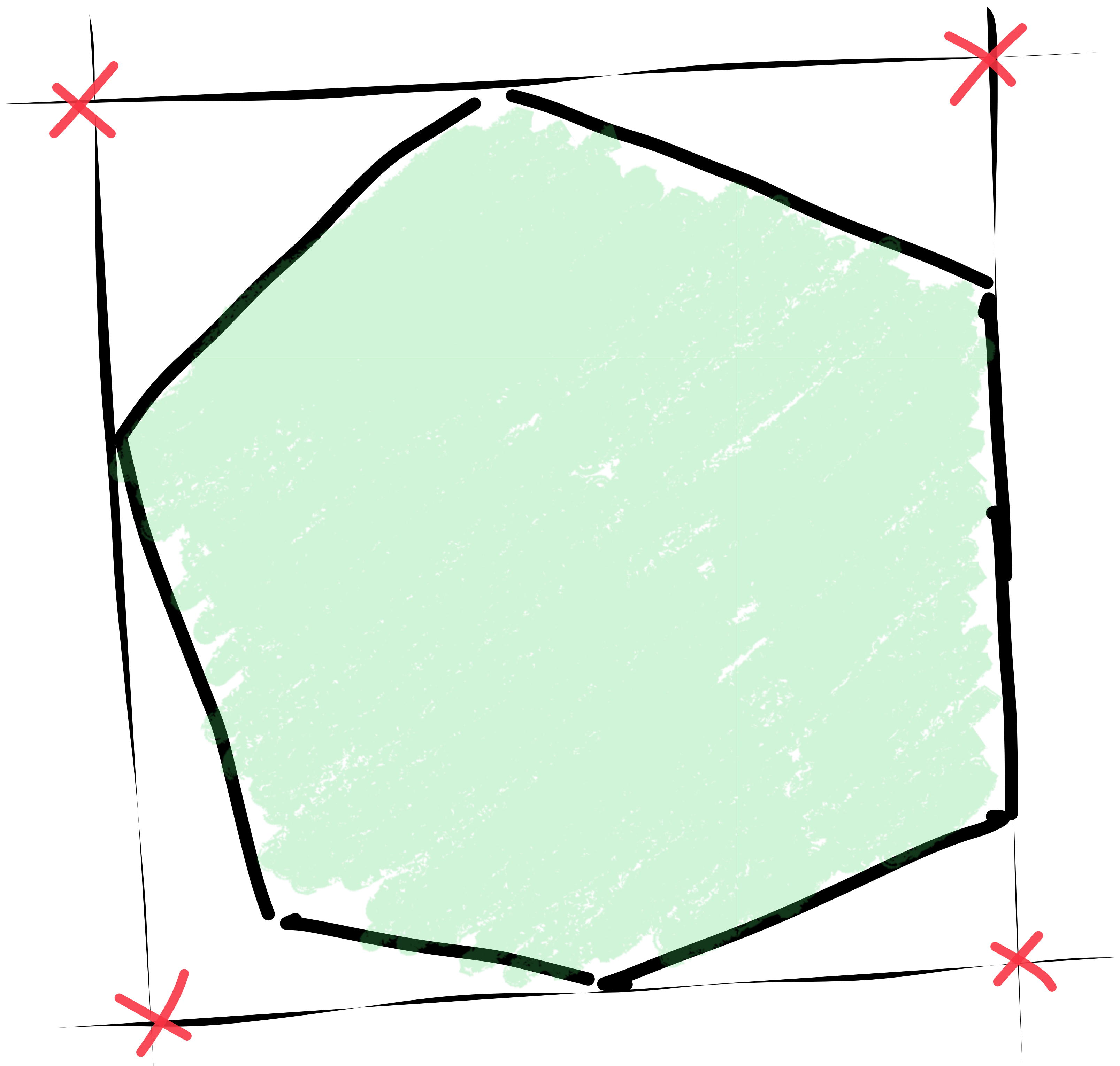

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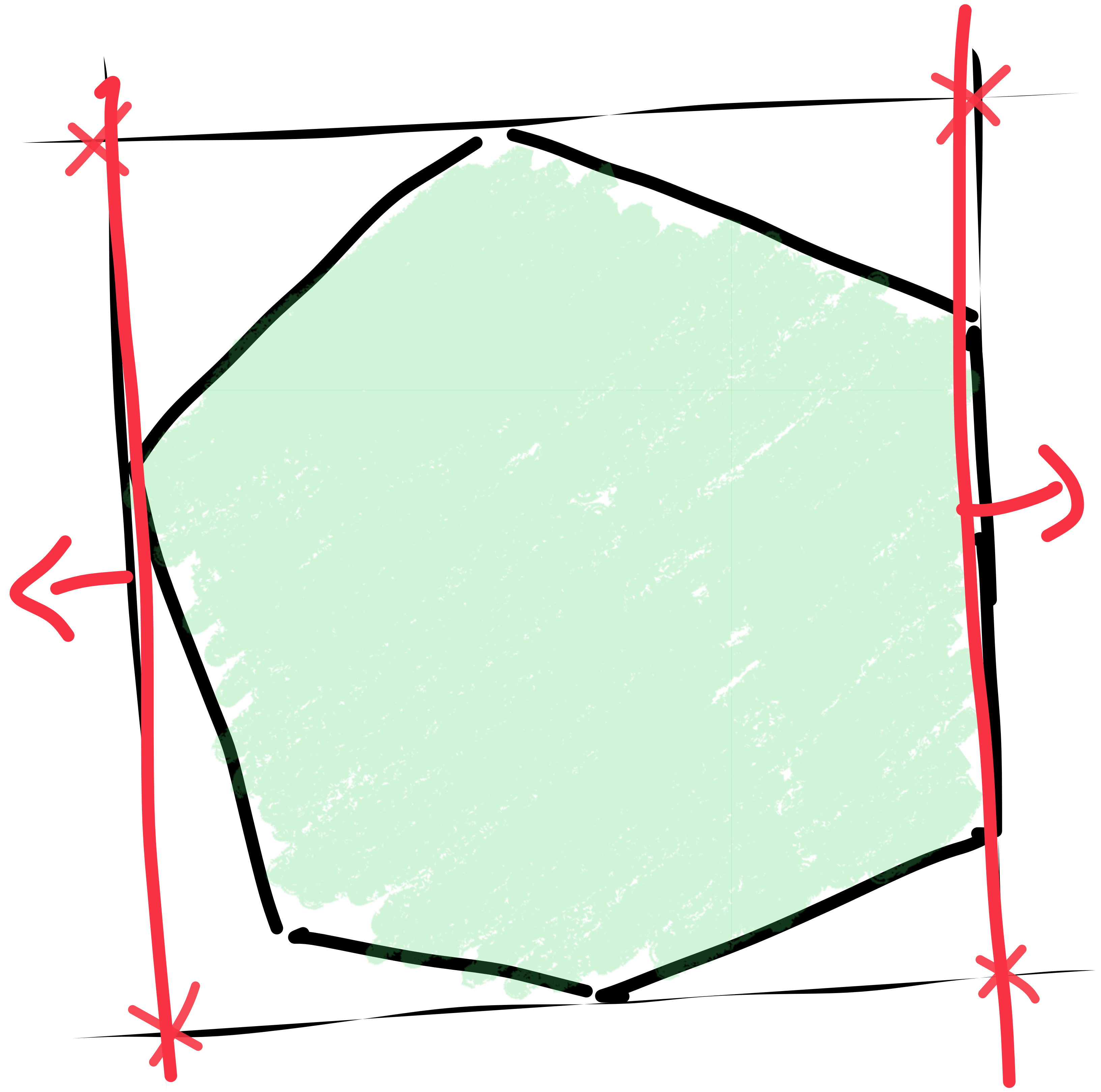
$$
\begin{array}{ccc}
 & P & \\
\swarrow^{x_1 \leq 0} & & \searrow^{x_1 \geq 1}
\end{array}
$$



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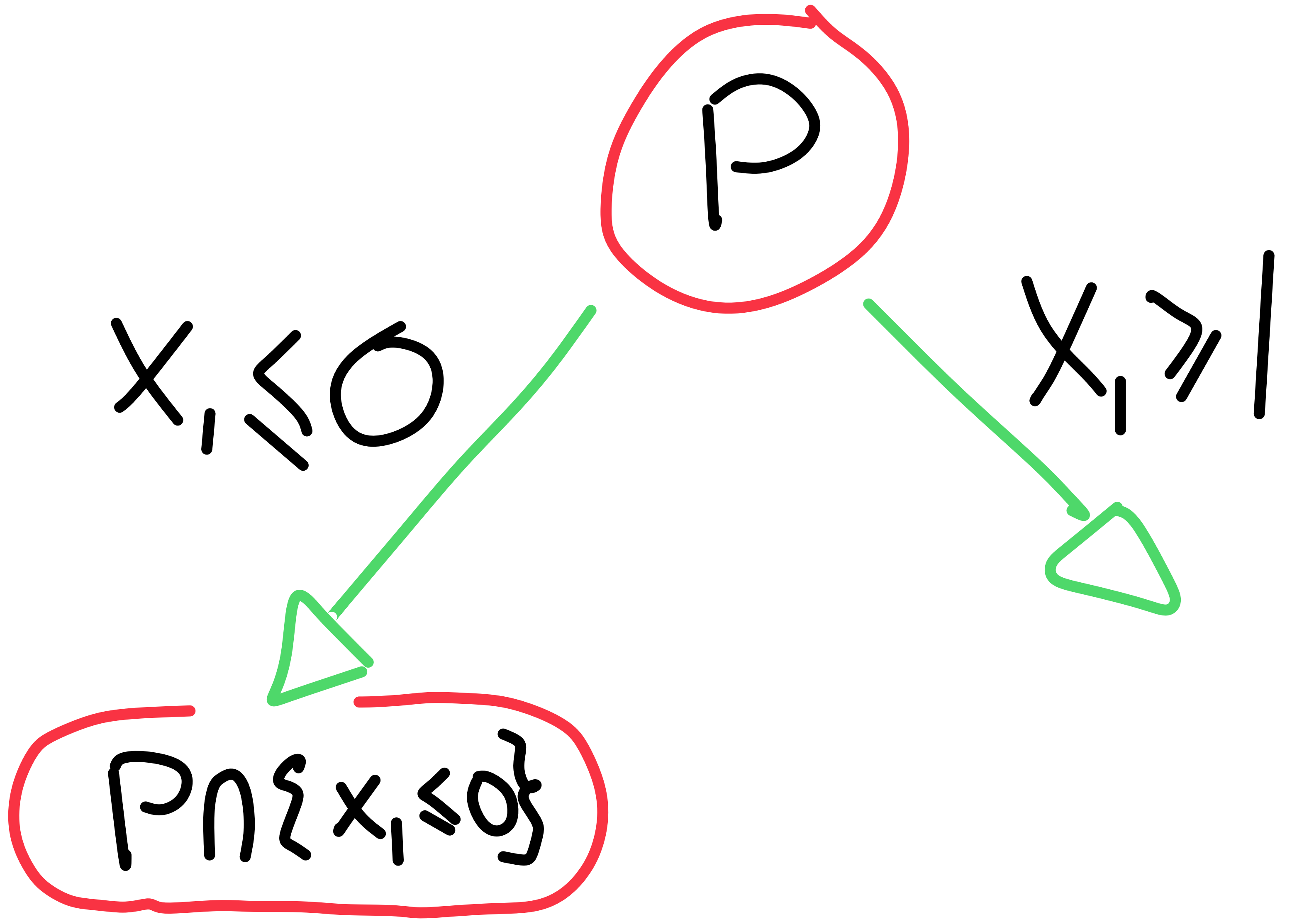

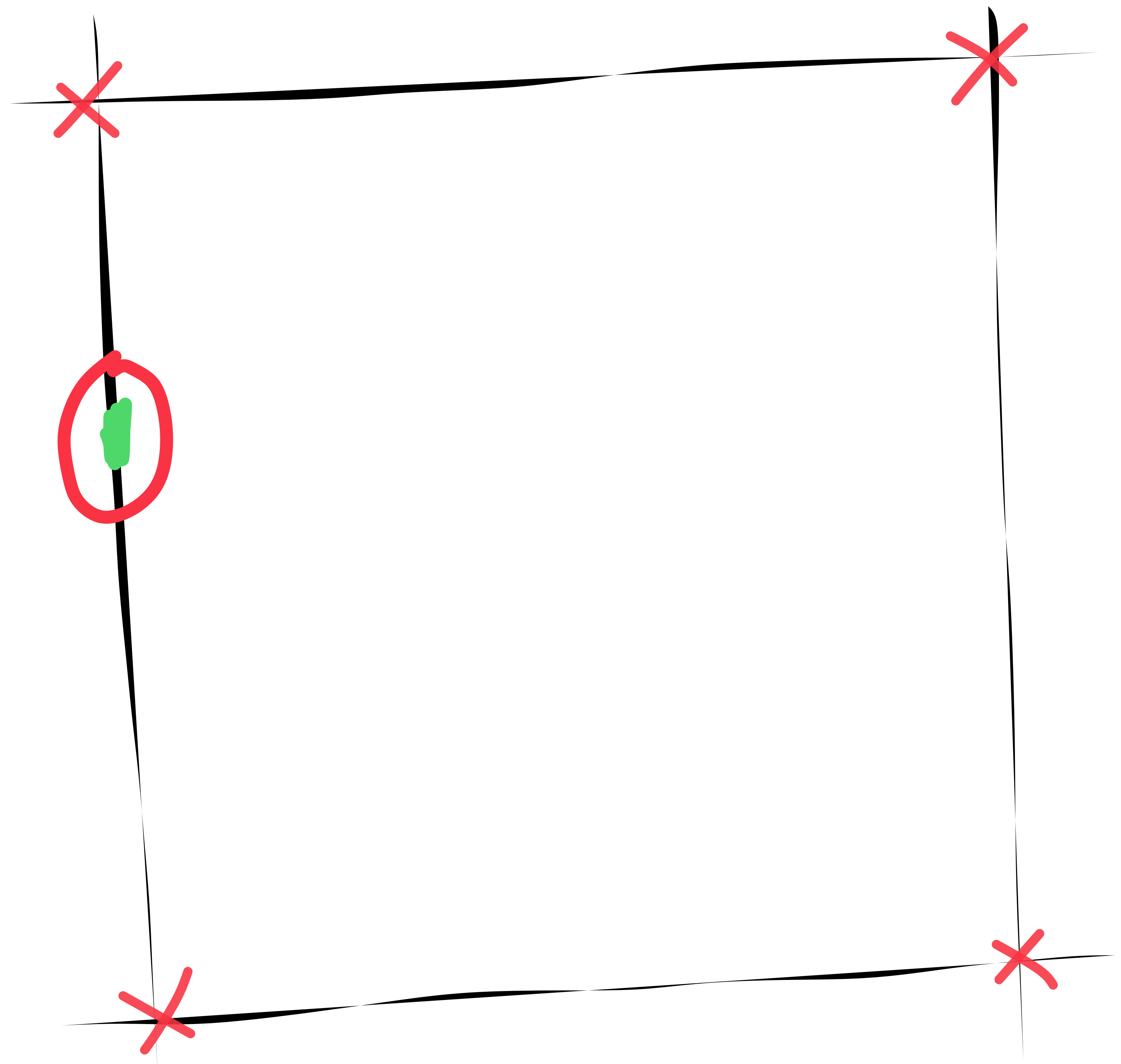

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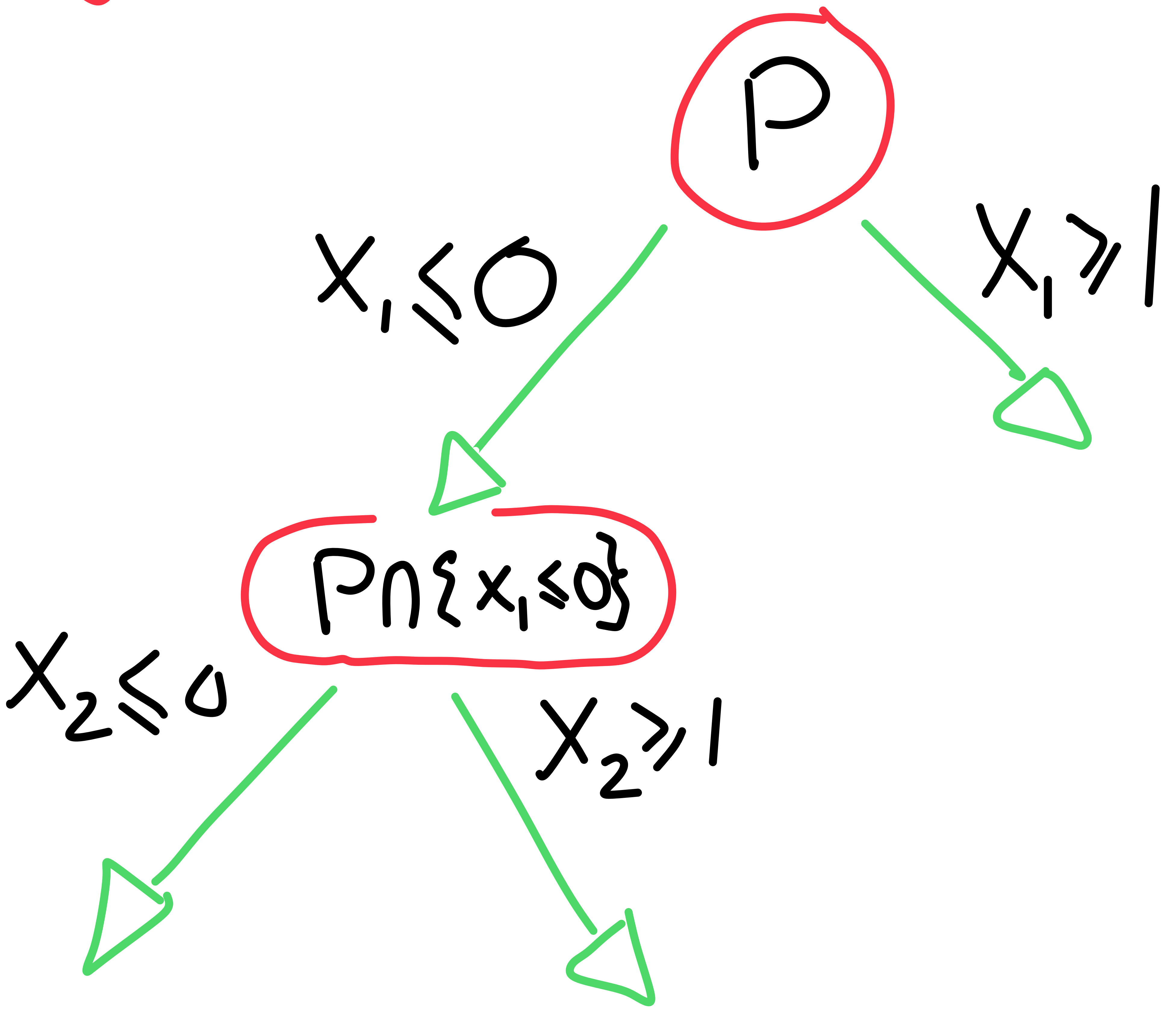

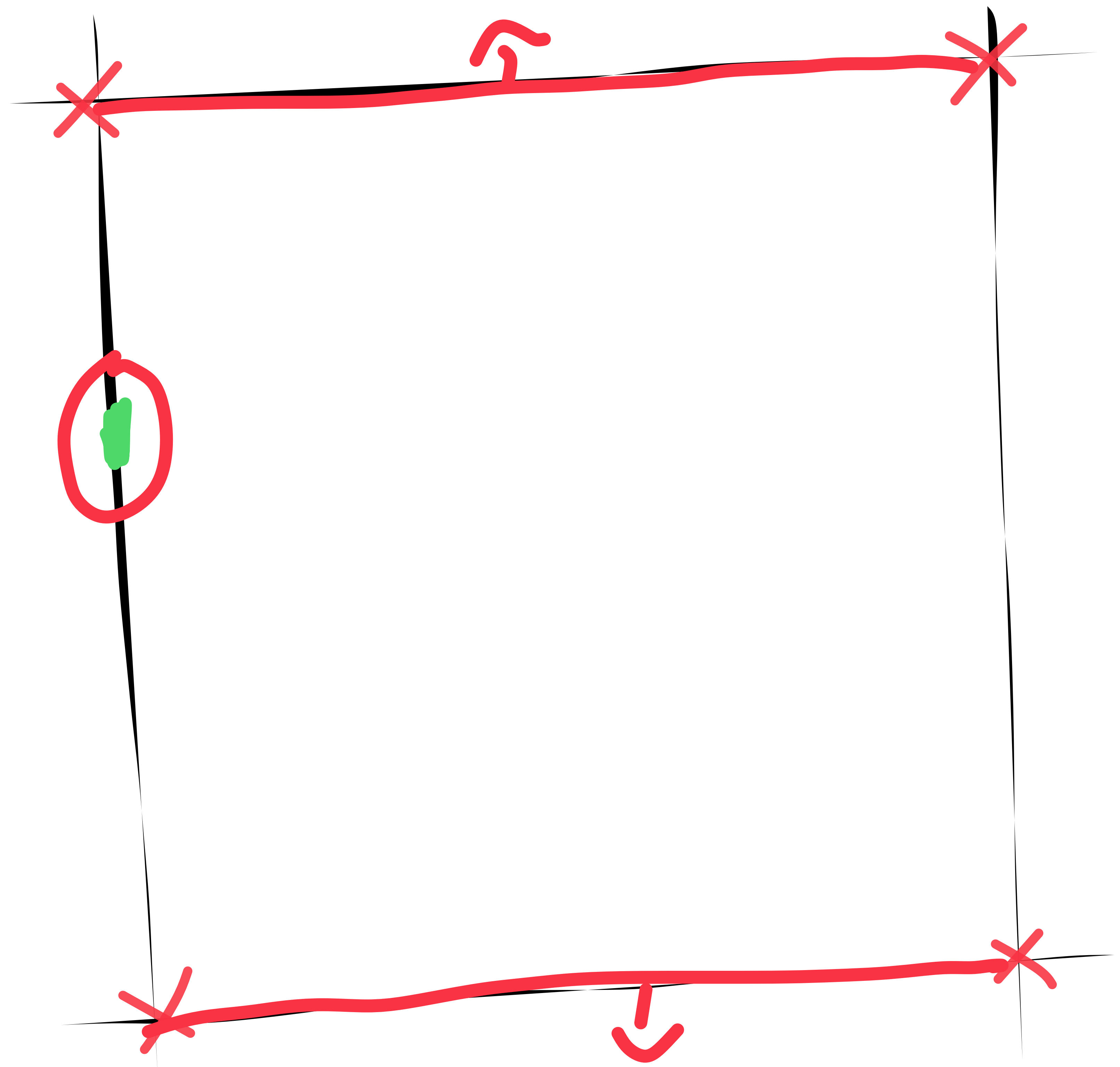

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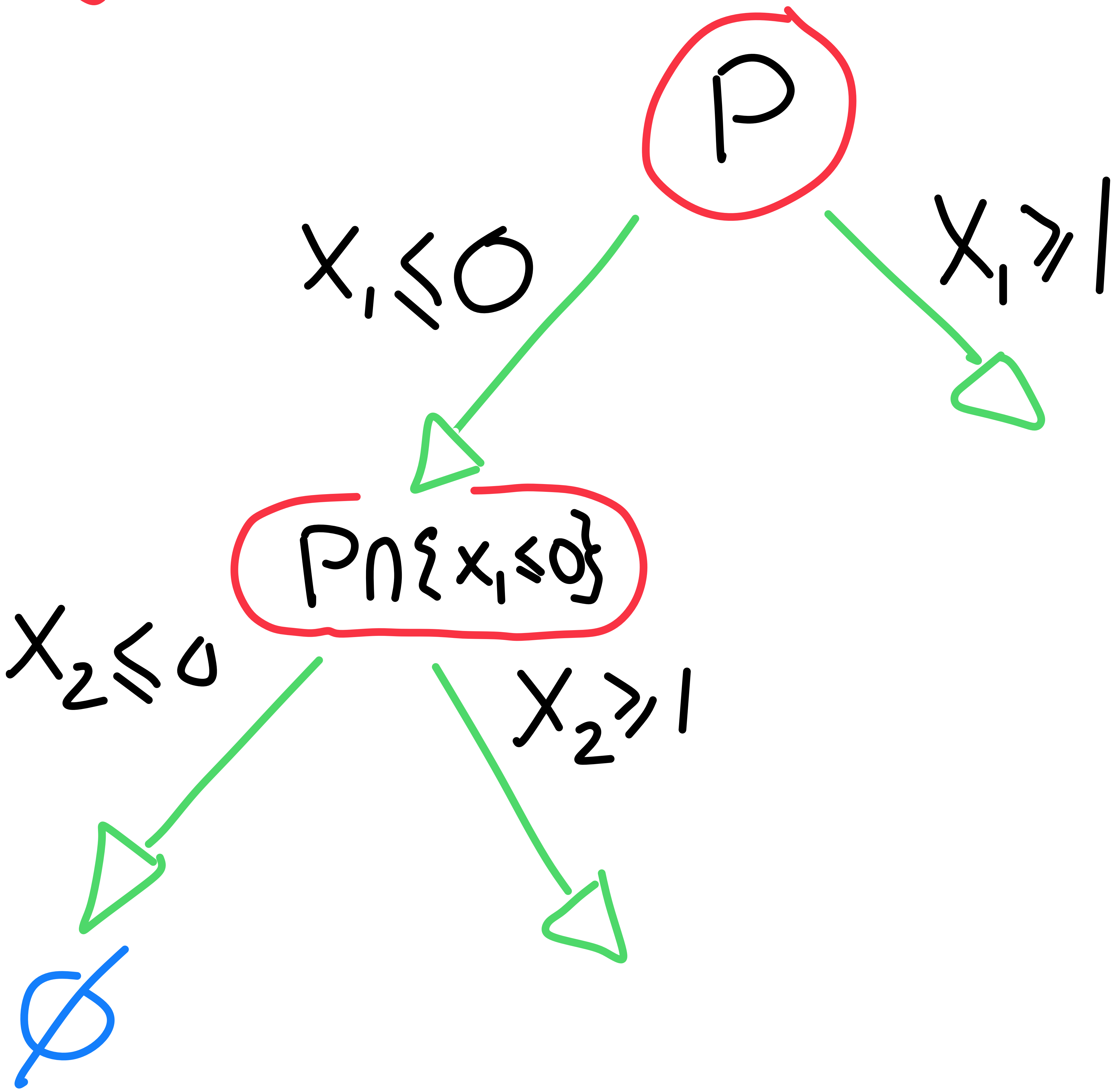

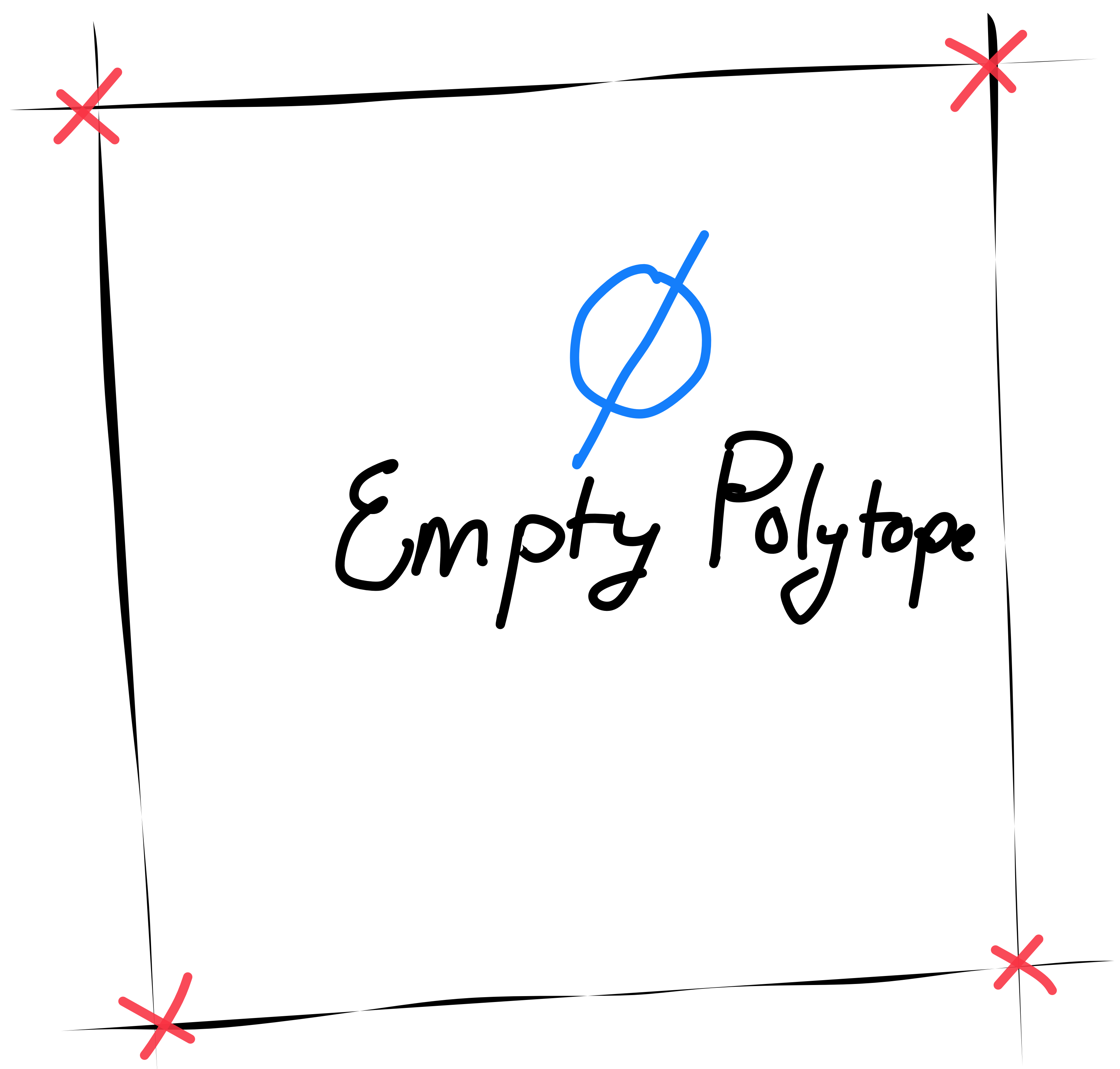

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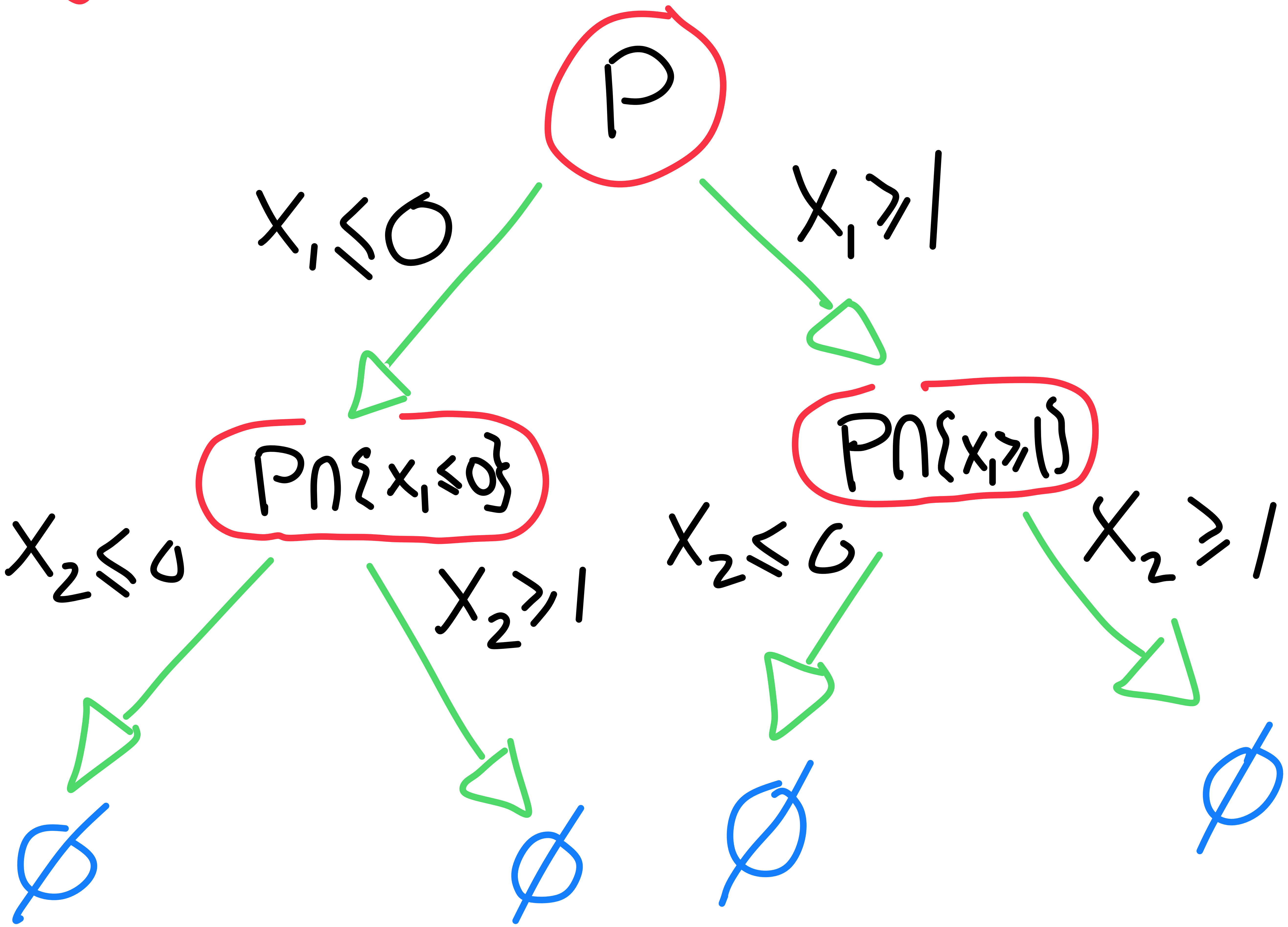

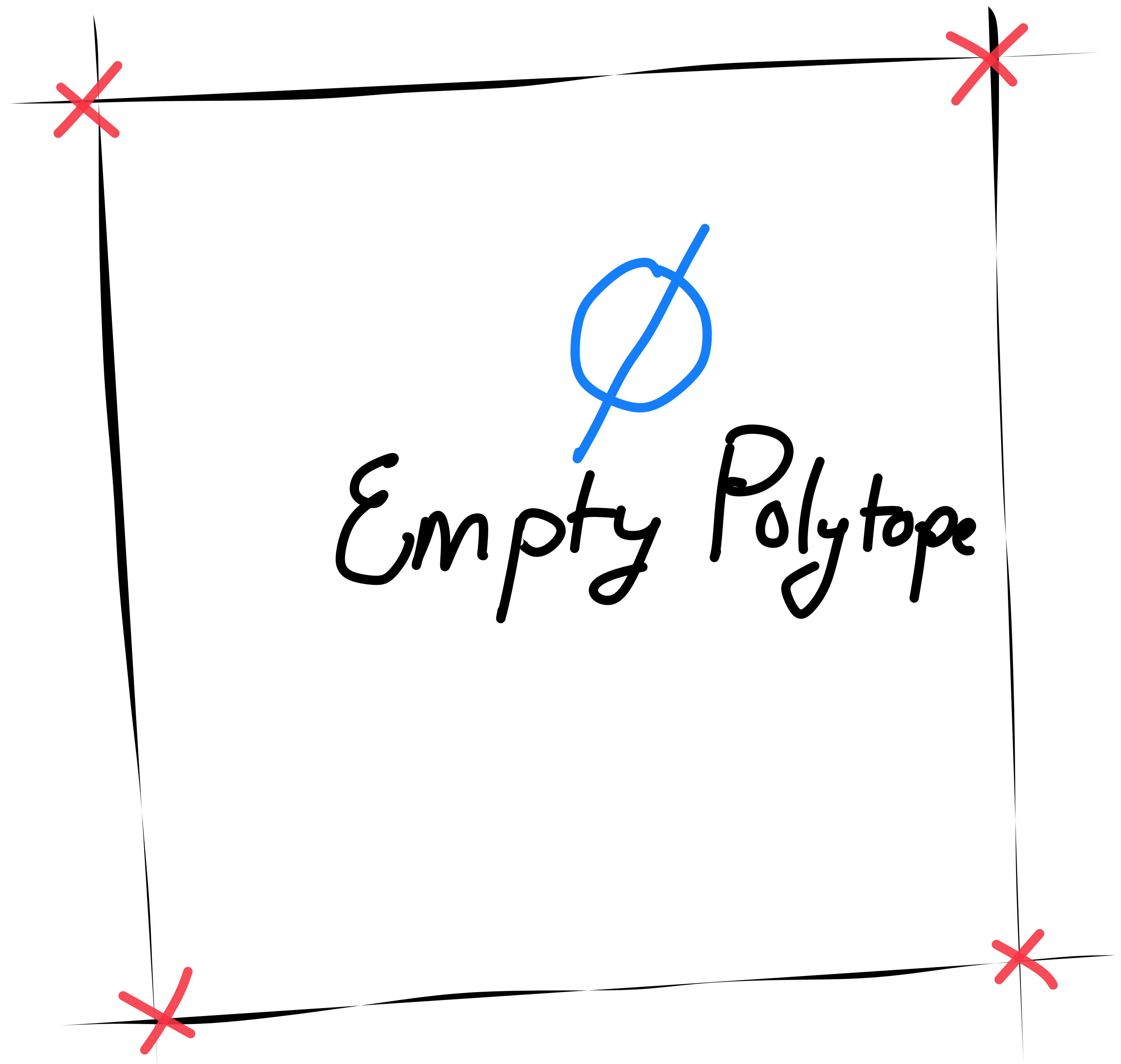

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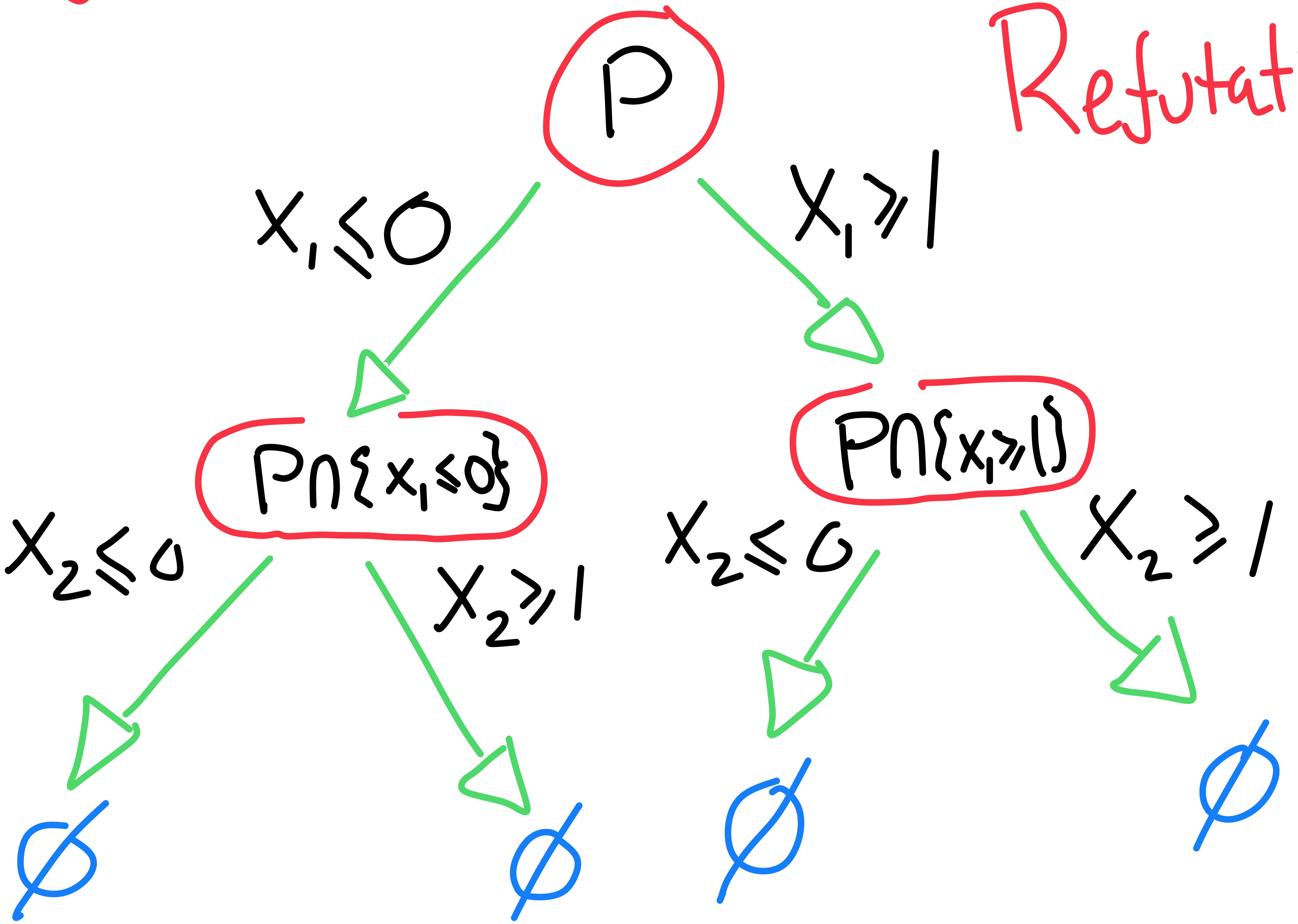
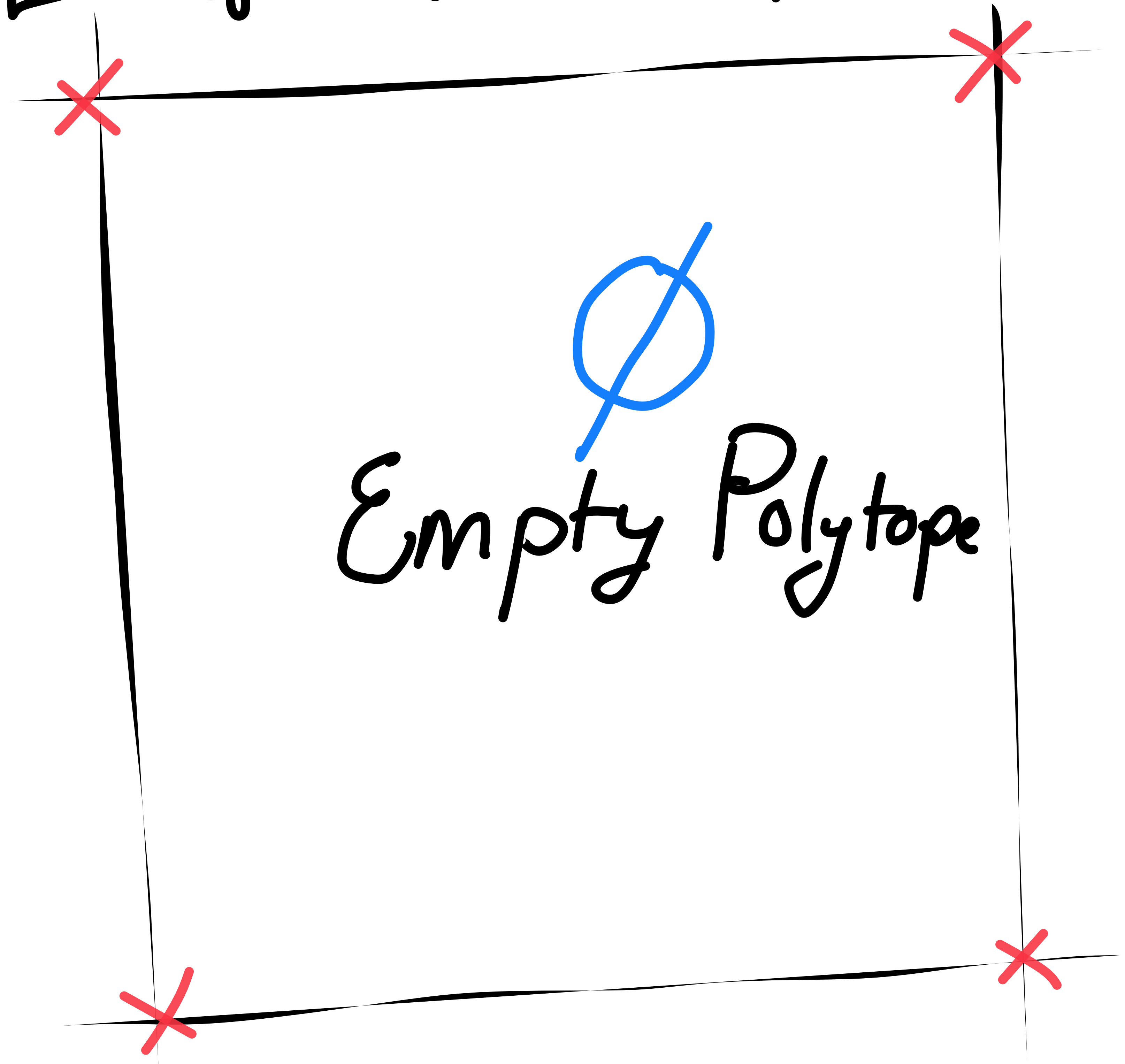


Refutation!

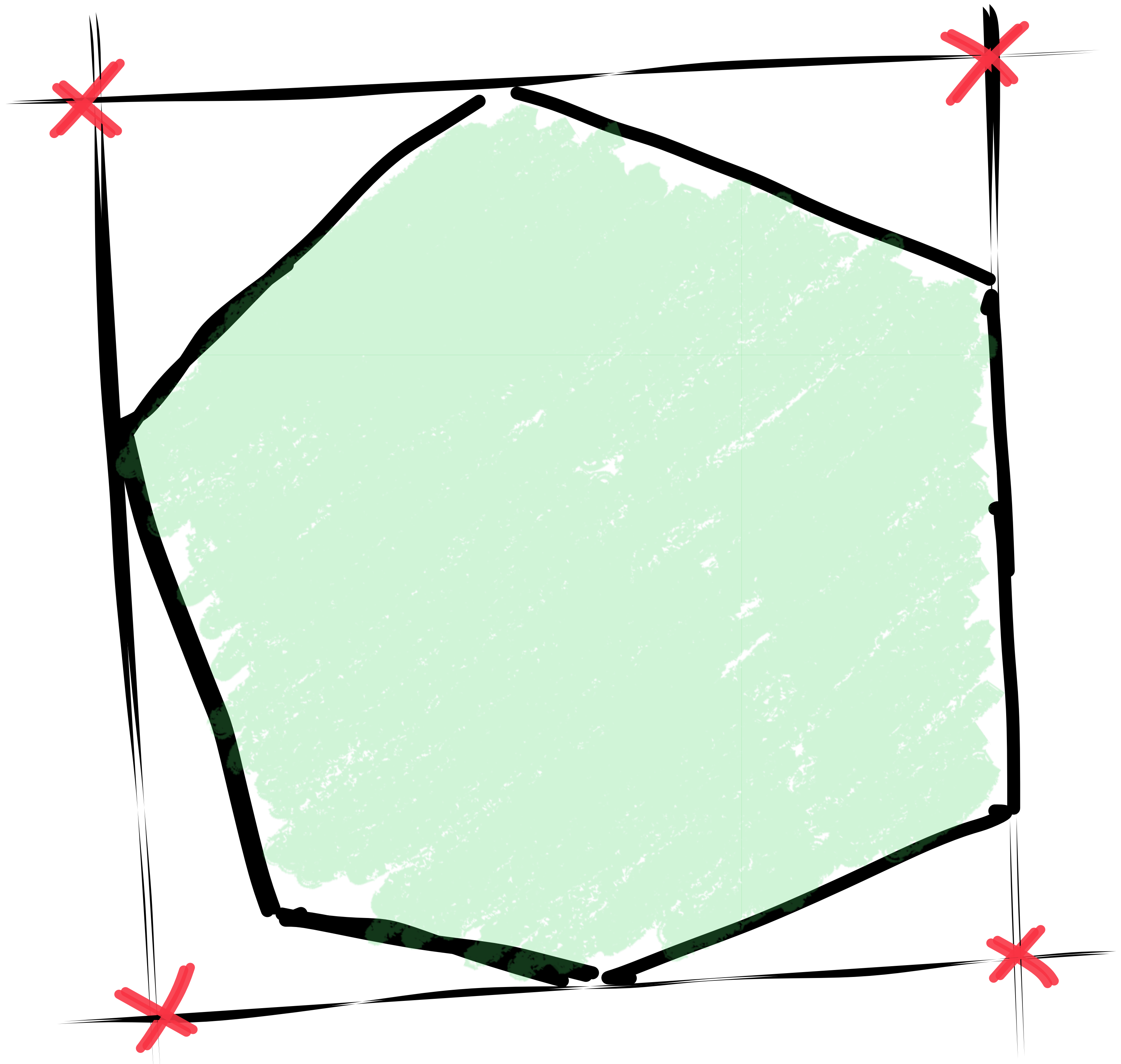
$\emptyset$ Derived at every Leaf of the tree



Stabbing Planes

$P = \{Ax \geq b\}$ s.t. $P \cap \mathbb{Z}^n = \emptyset$.

P



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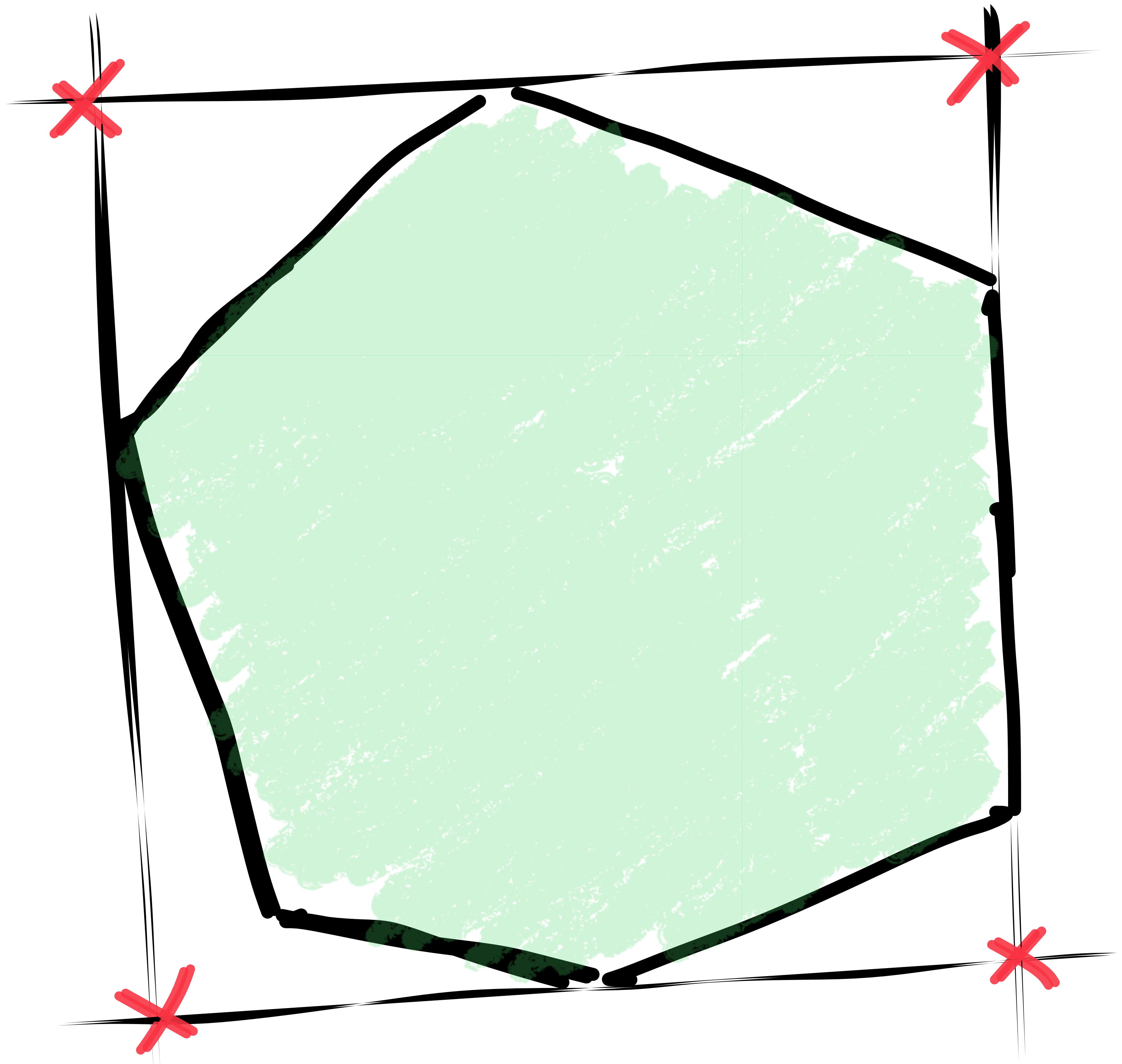
"query arbitrary linear inequalities"



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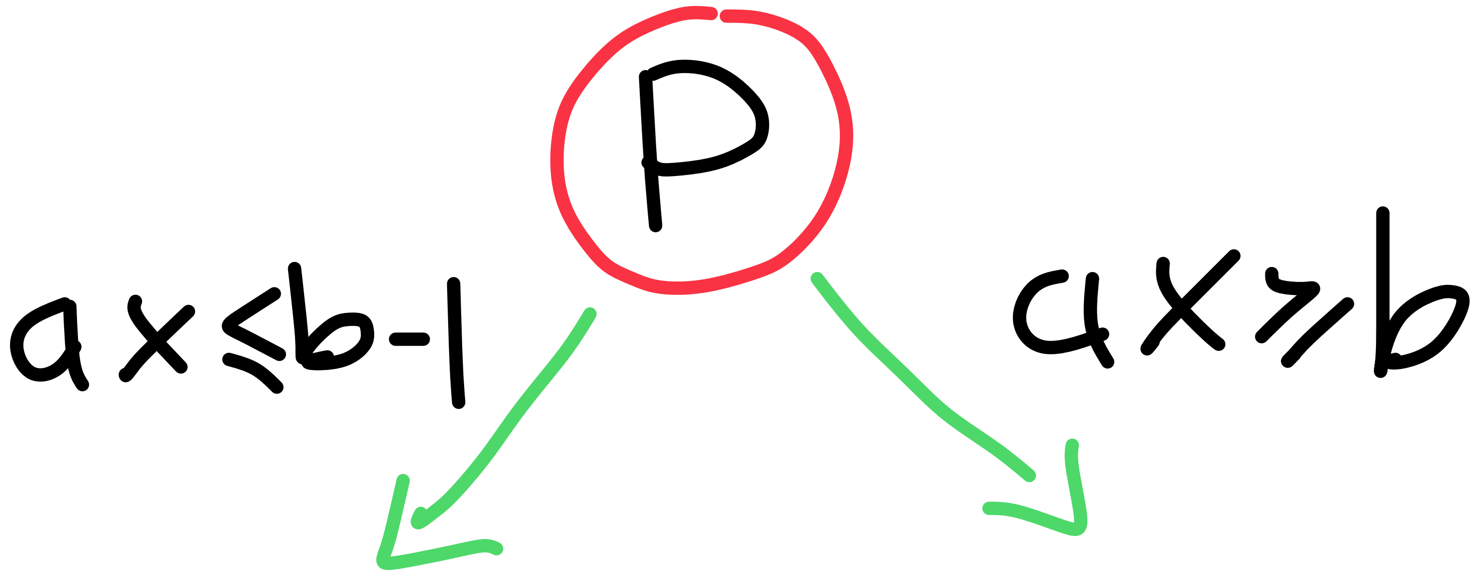

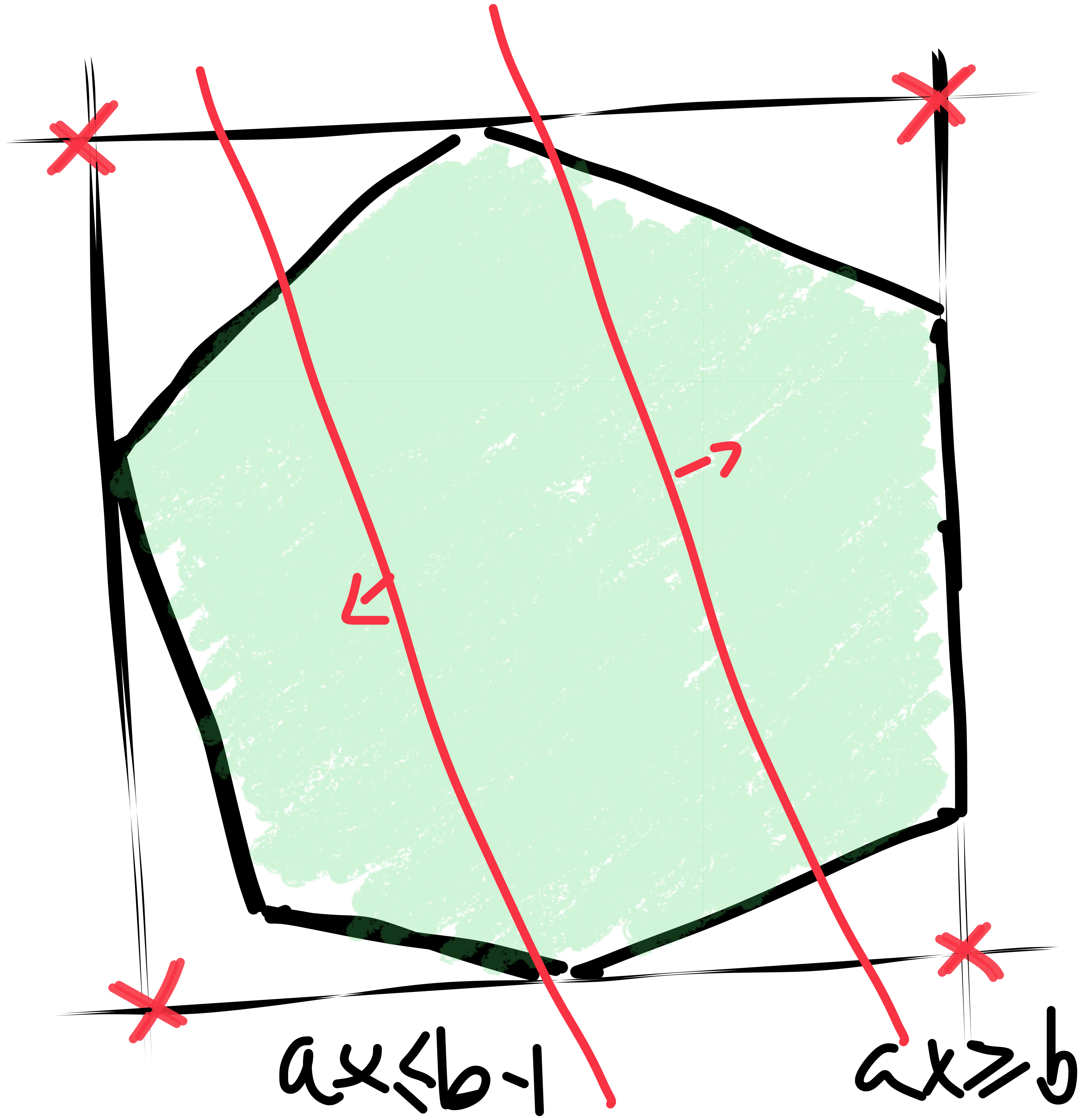

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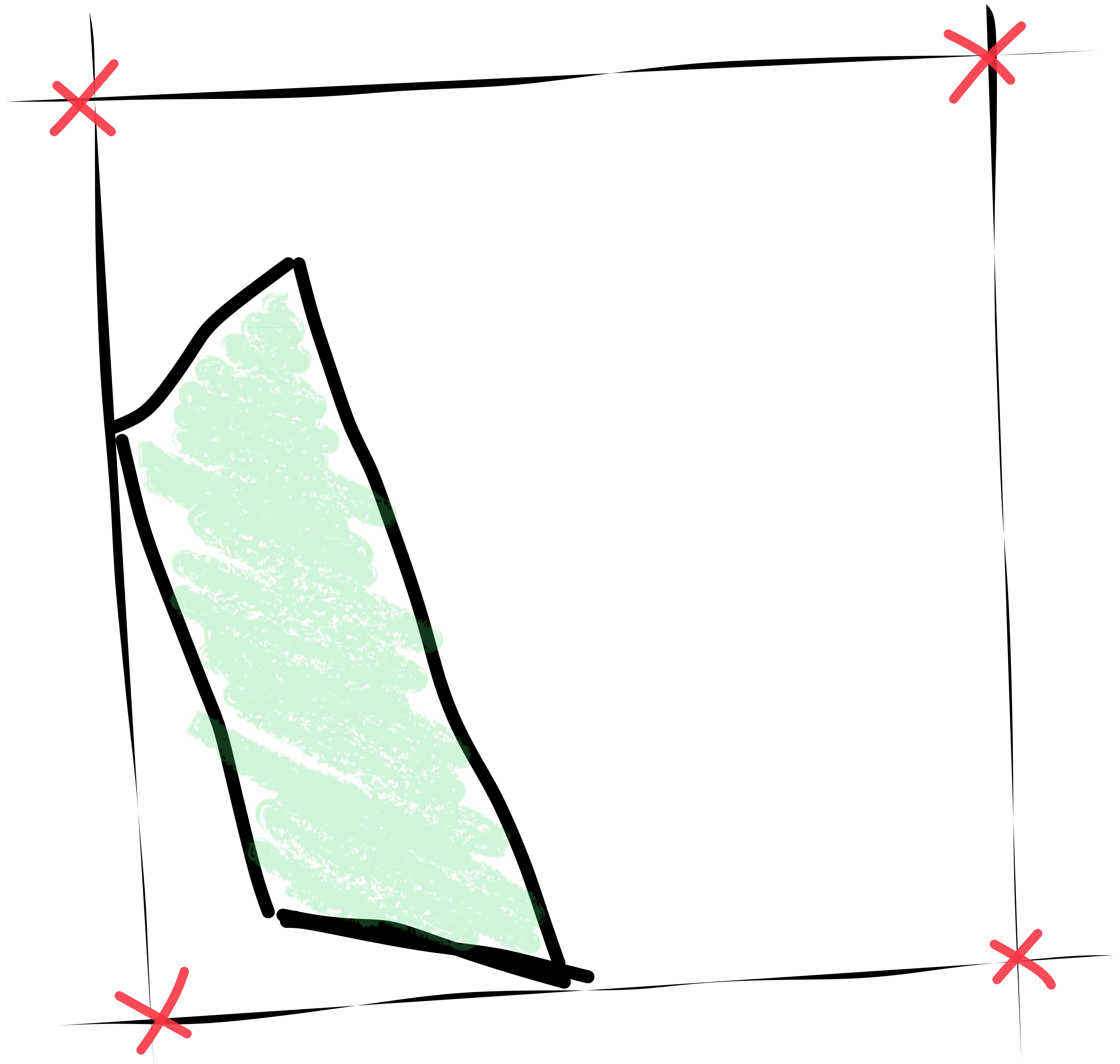
$$\begin{tikzcd} & P \arrow[dl, "ax \leq b-1"'] \arrow[dr, "ax \geq b"] & \\ \boxed{P \cap \{ax \leq b-1\}} & & {} \end{tikzcd}$$



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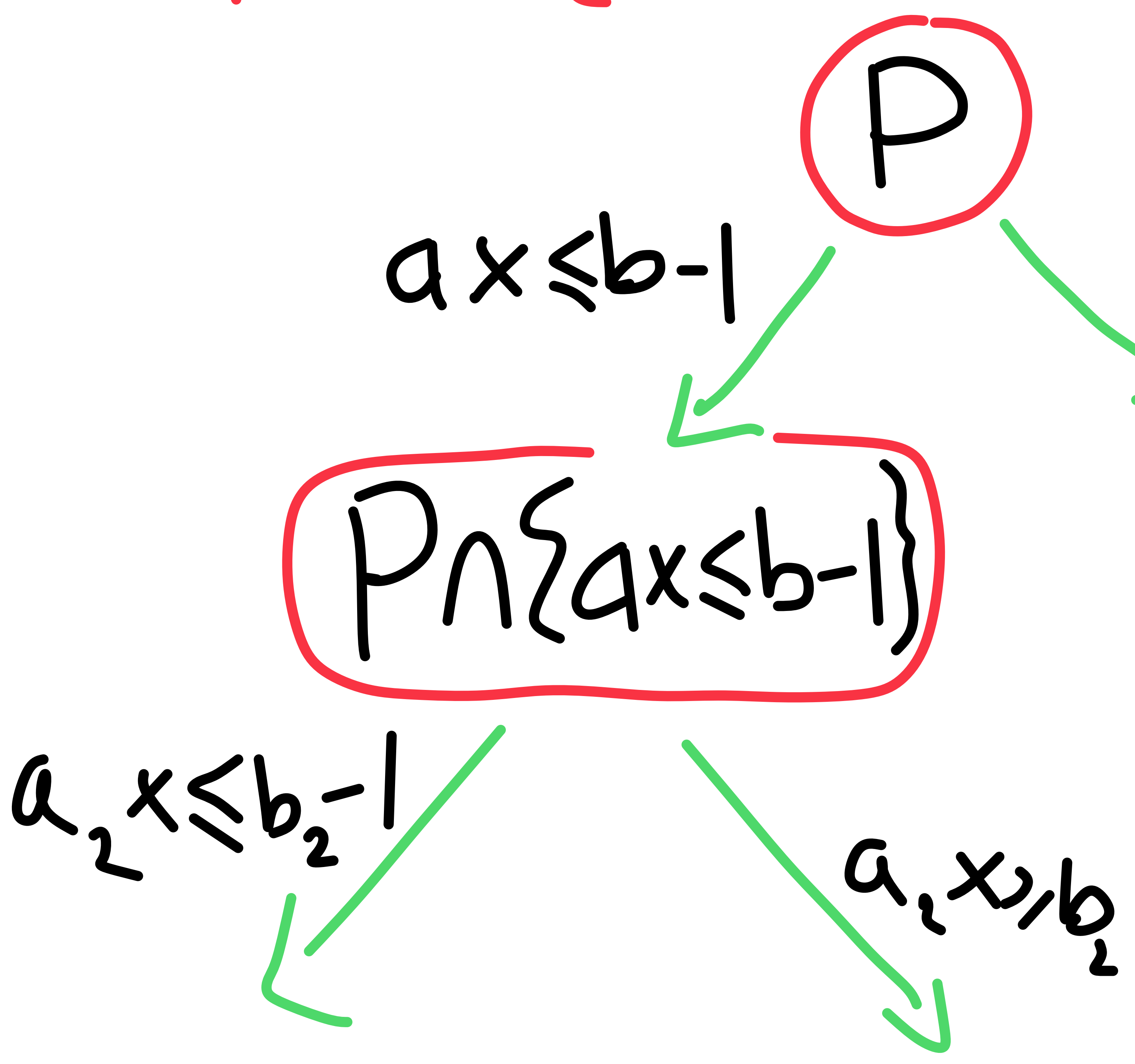

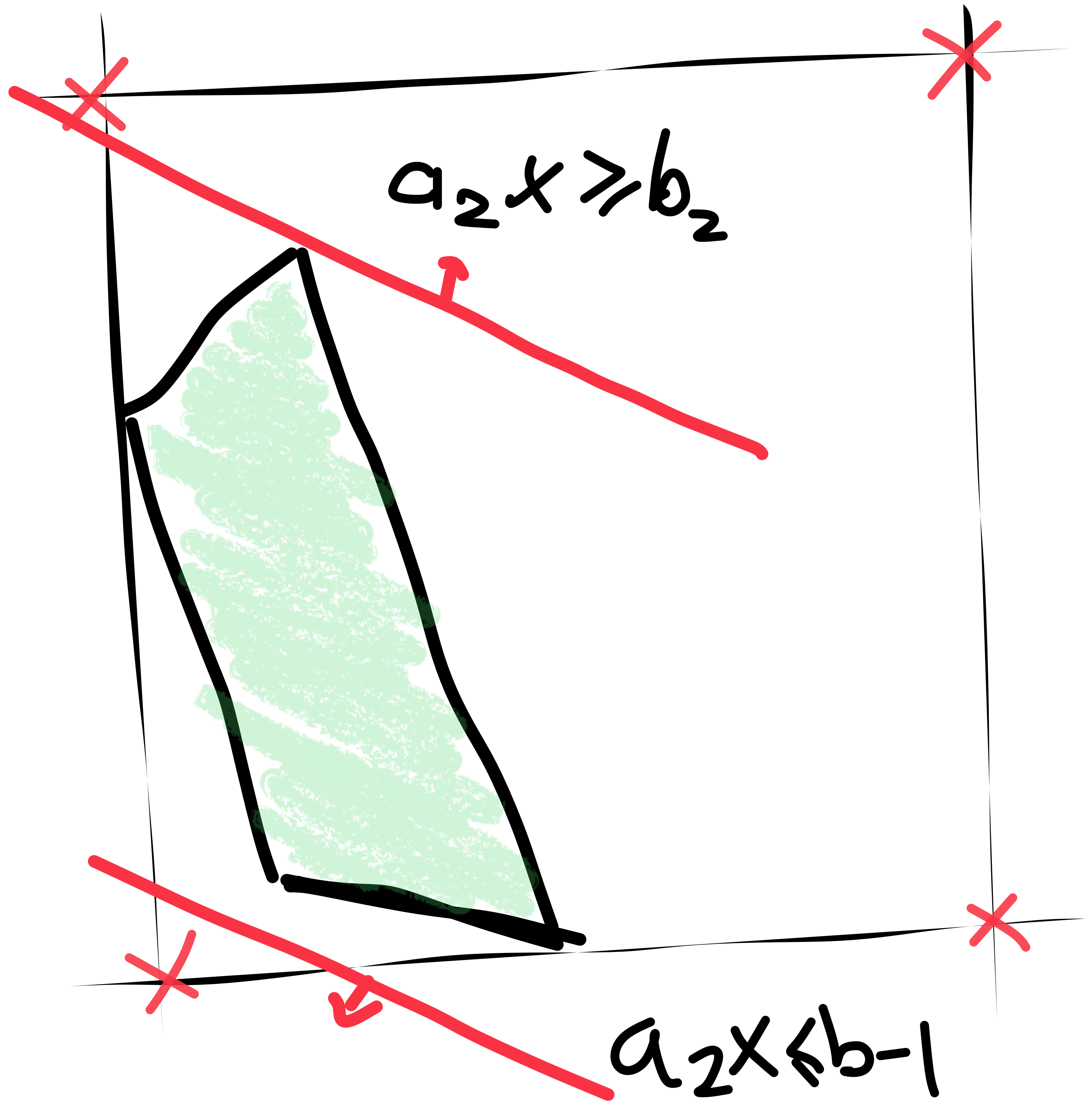

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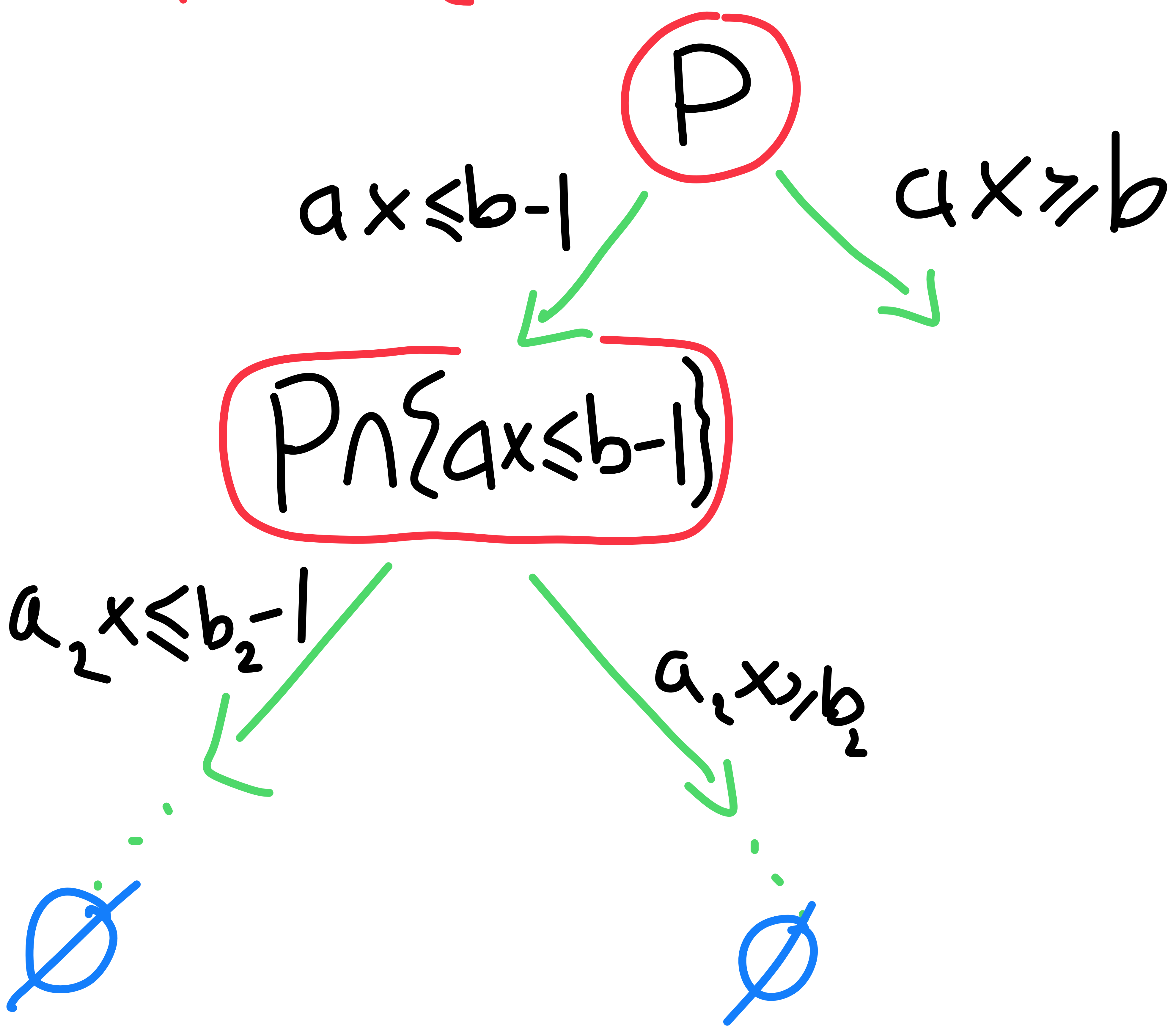

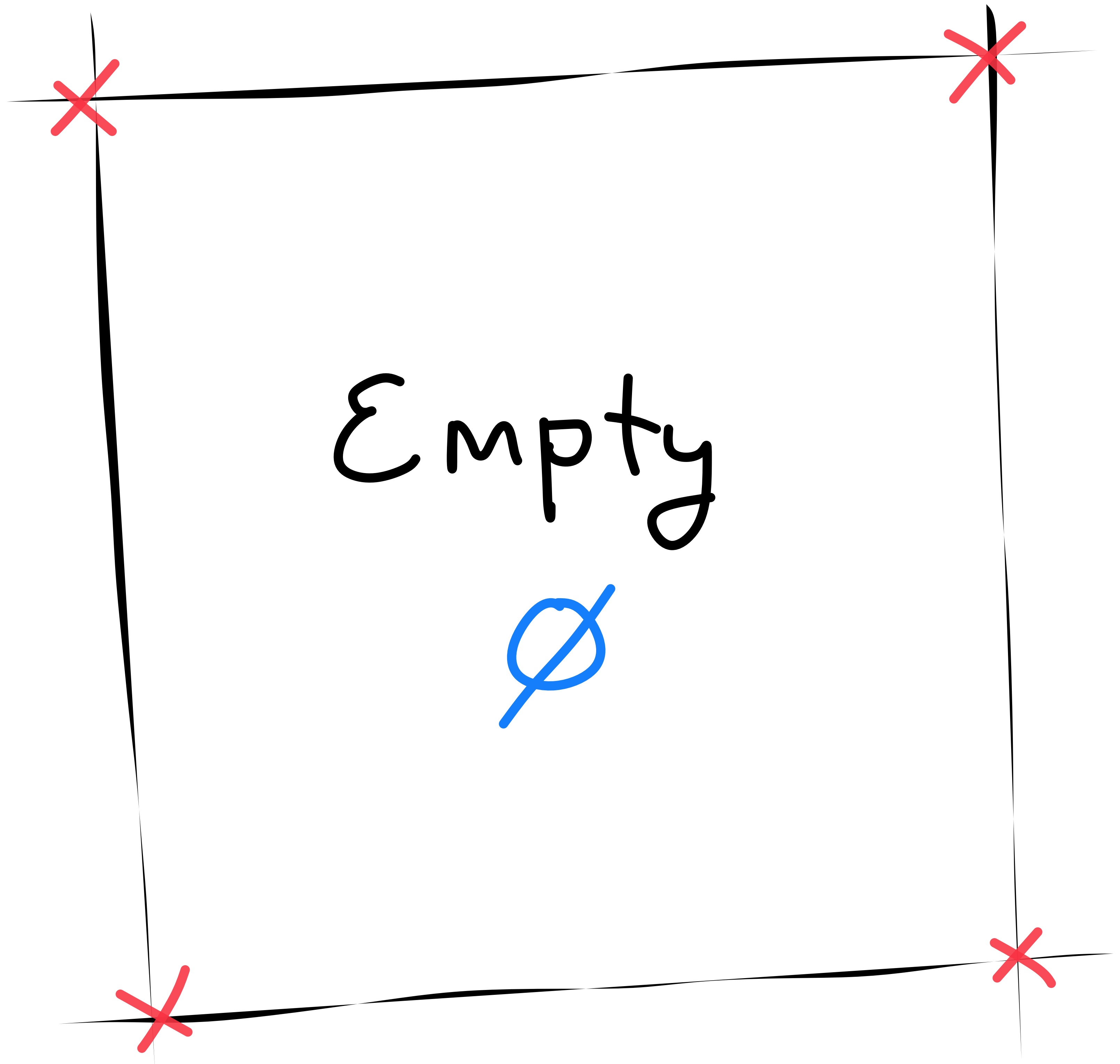

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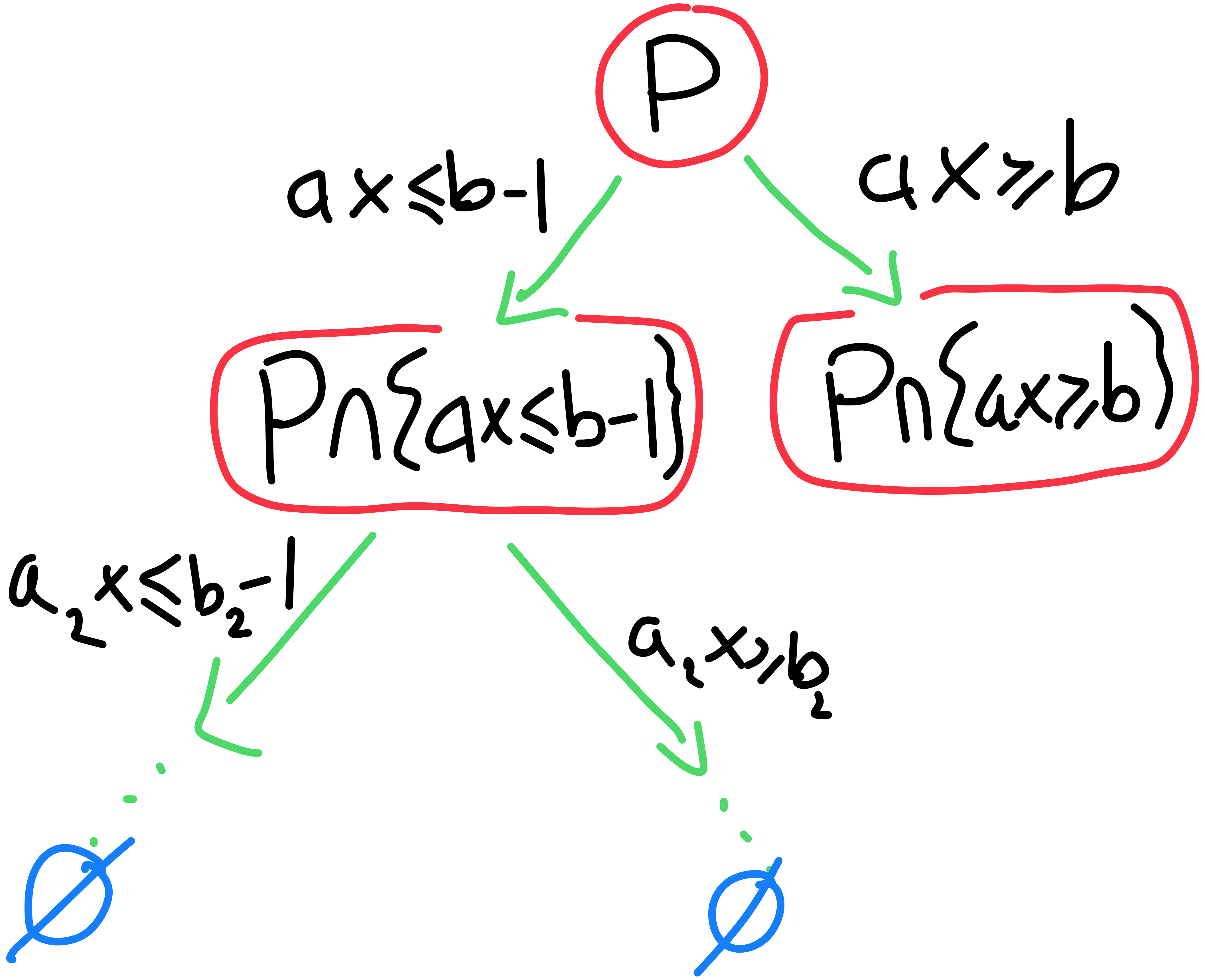

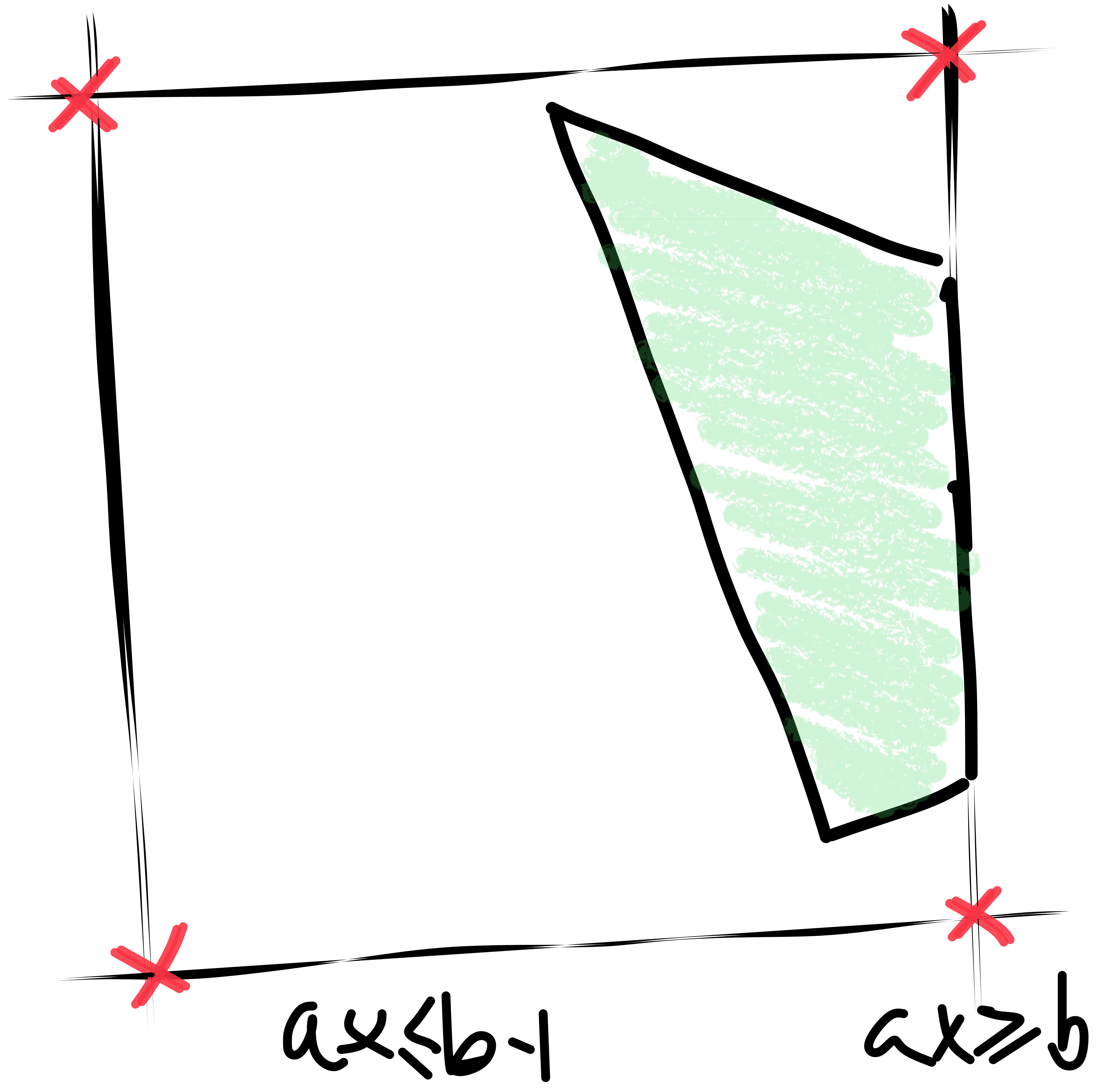

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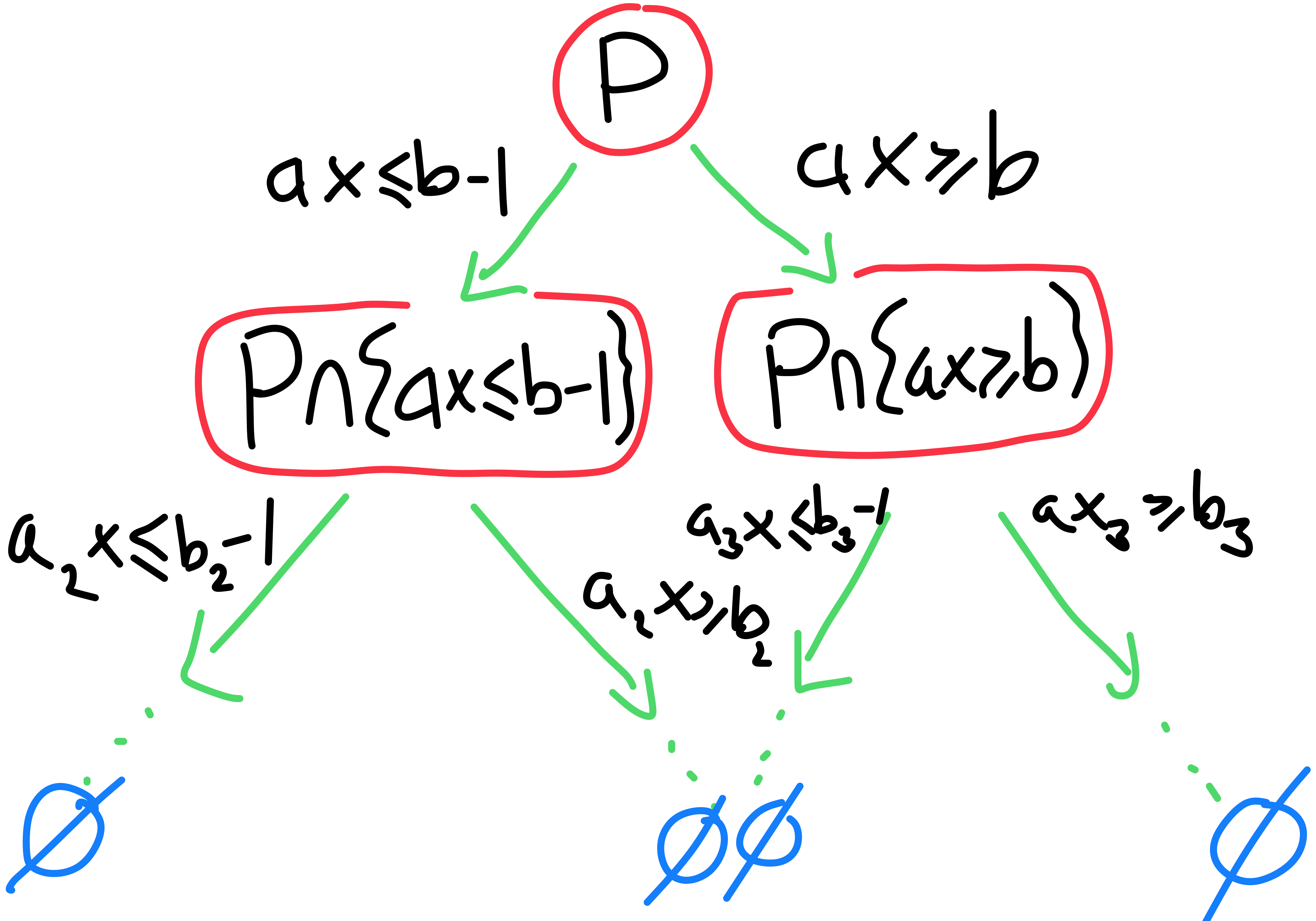

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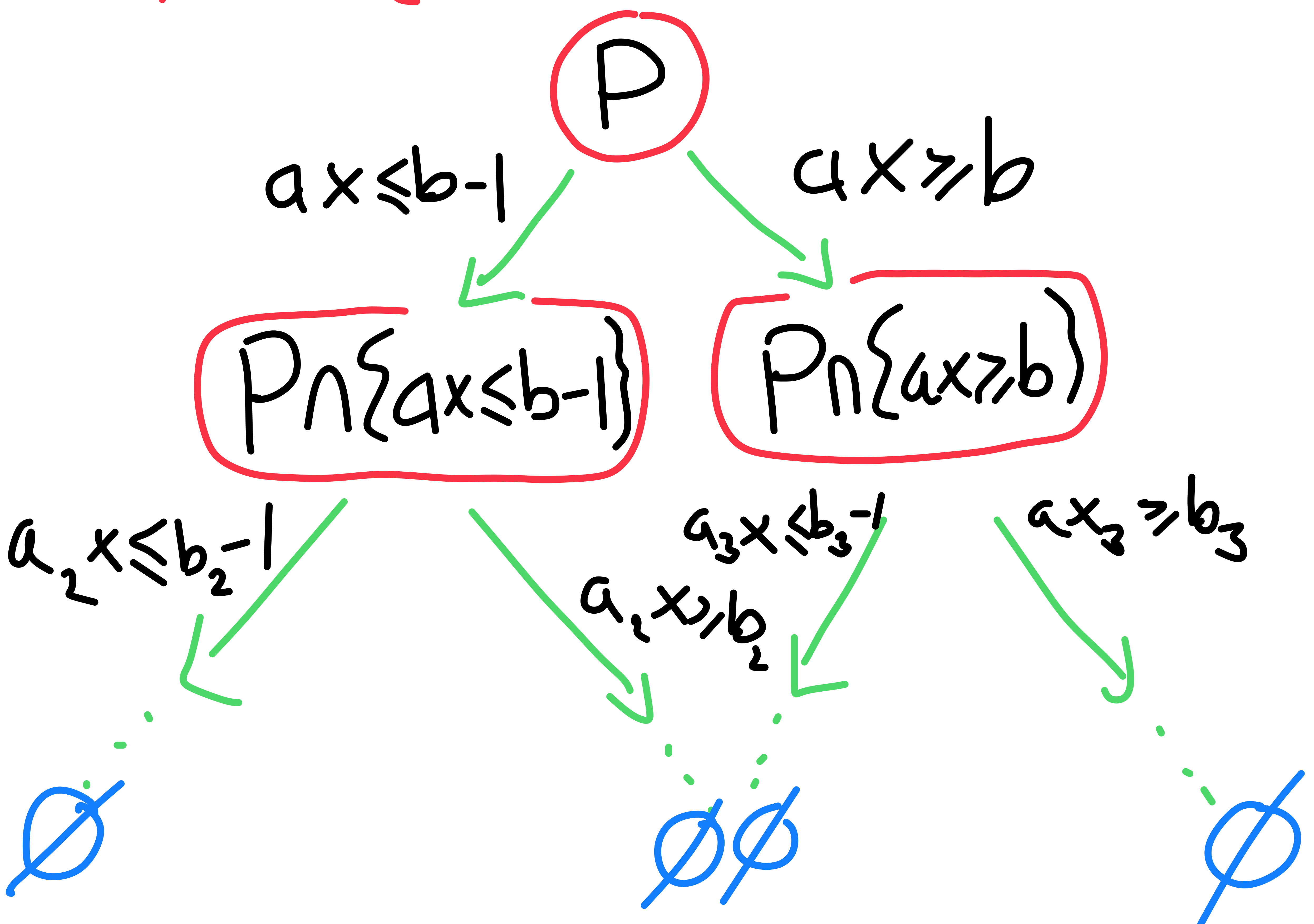

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Empty polytope $\emptyset$ deduced at every leaf proves $P \cap \mathbb{Z}^n = \emptyset$

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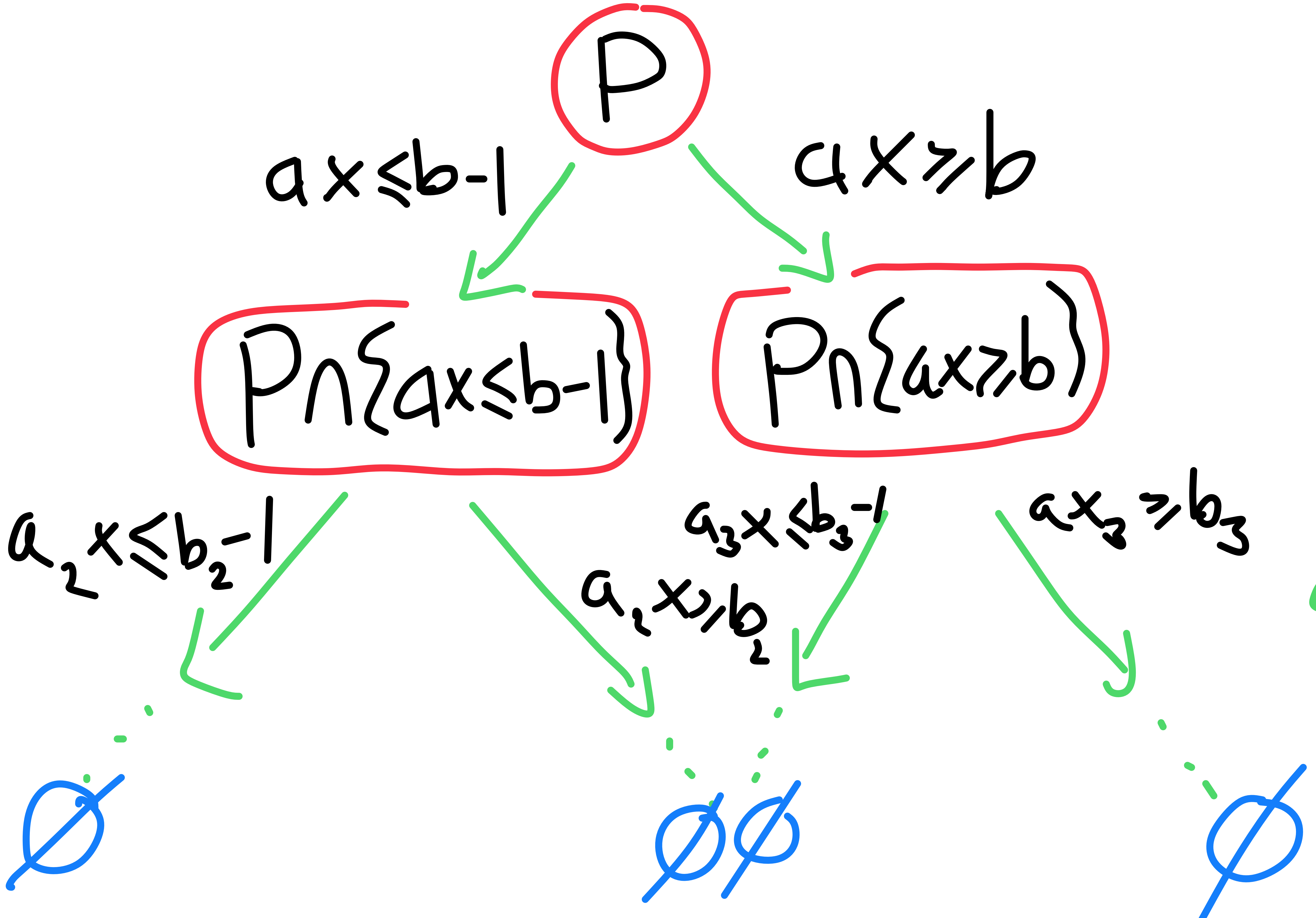

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Size: # of queries

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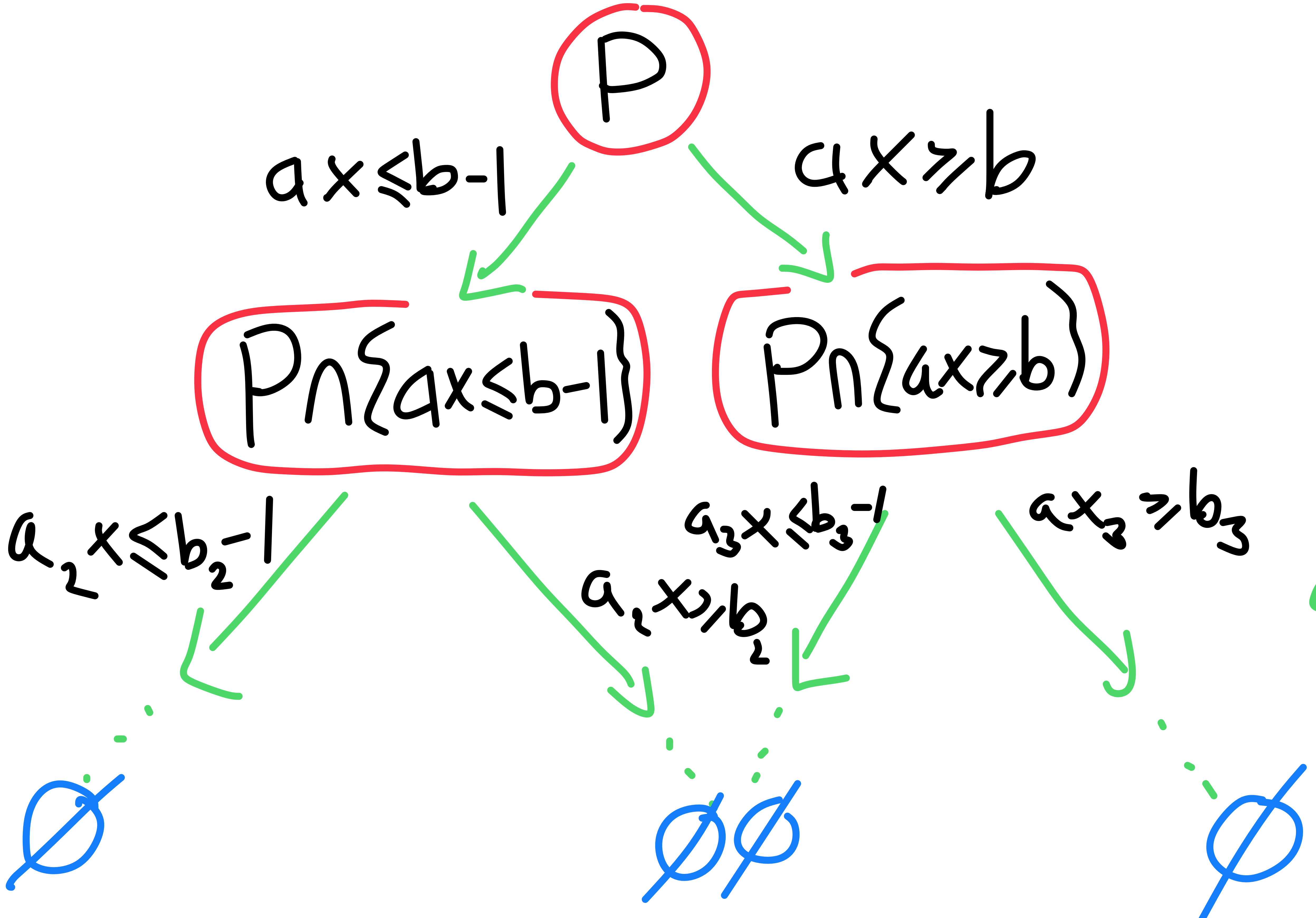

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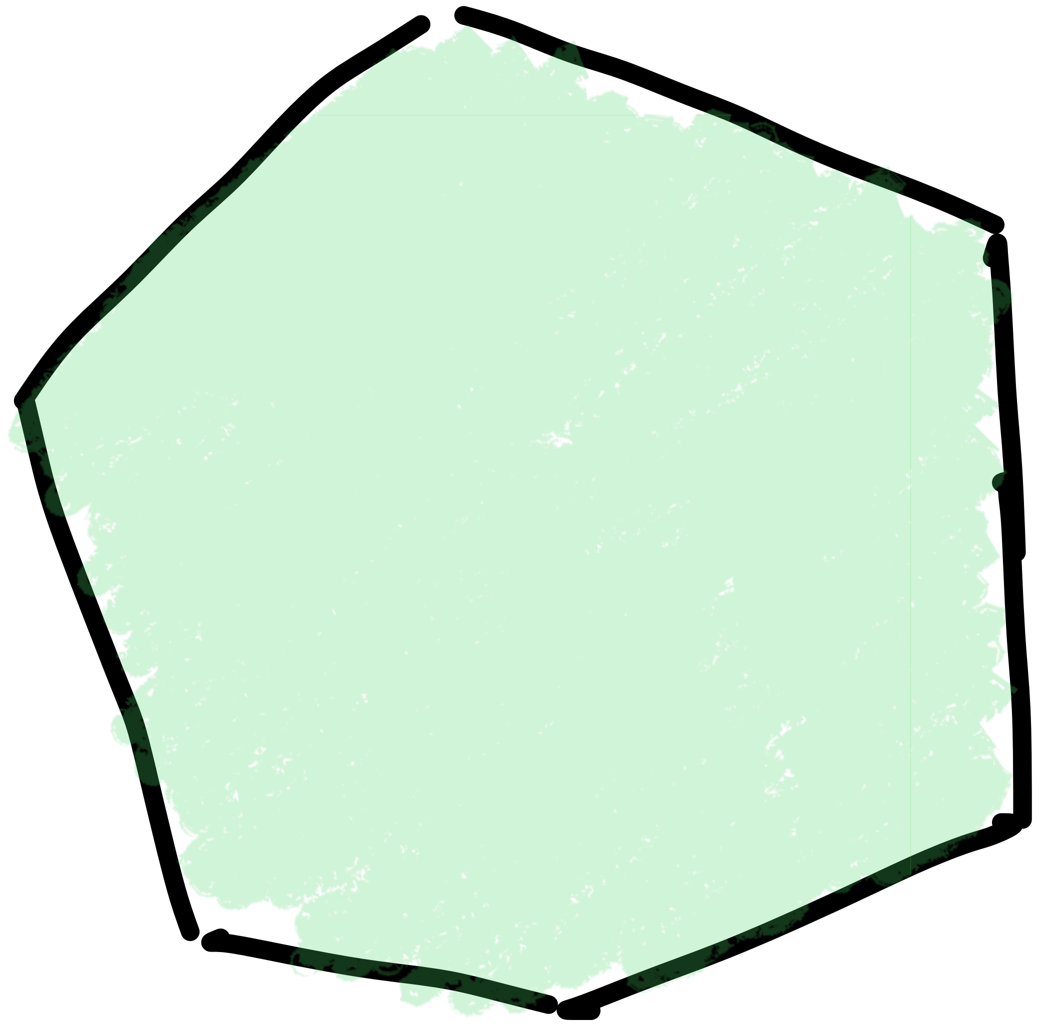
Size: # of queries $\approx$ bit-size [DT20]

Stabbing Planes vs Cutting Planes

- Branch-and-cut allows Cutting planes deductions

Stabbing Planes vs Cutting Planes

- Branch-and-cut allows Cutting planes deductions
  - Superfluous as SP can simulate CP



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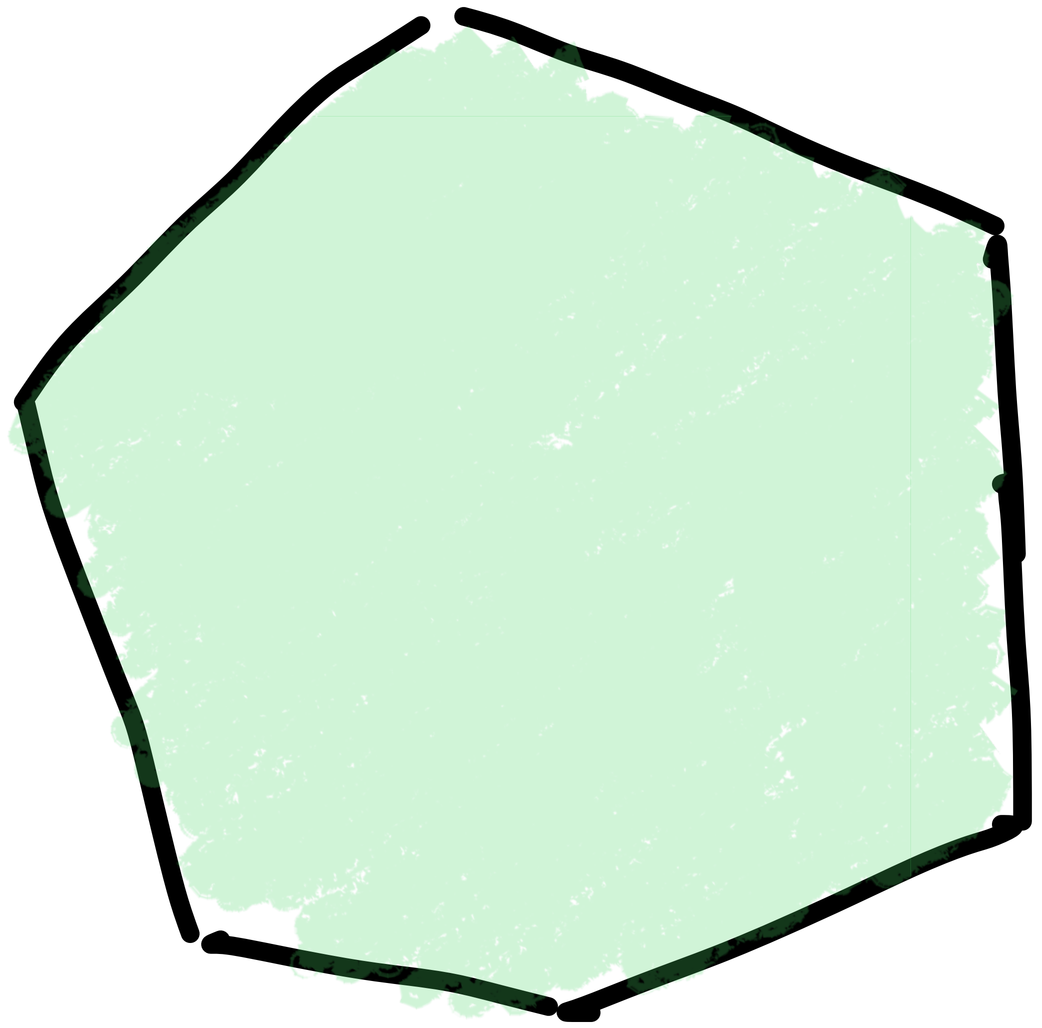
Clm: pathlike SP = CP



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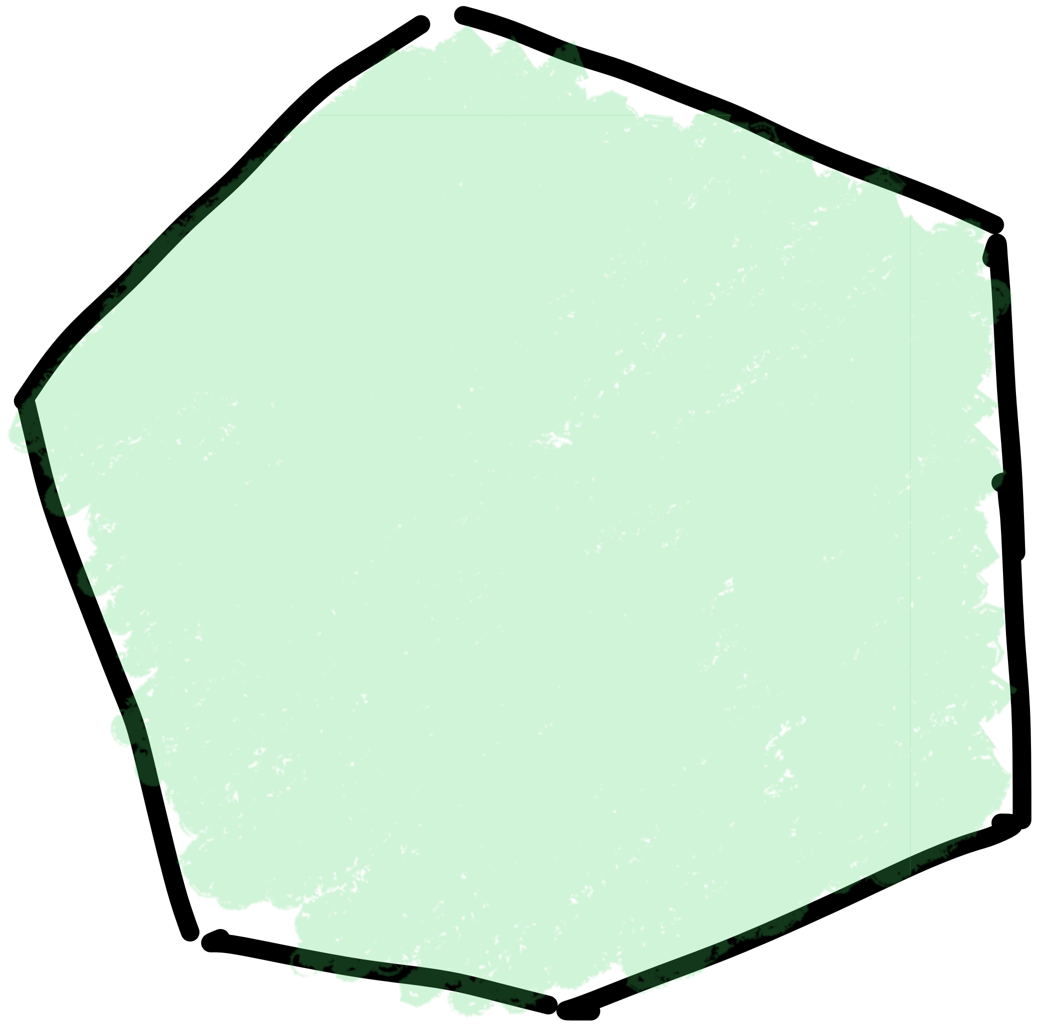

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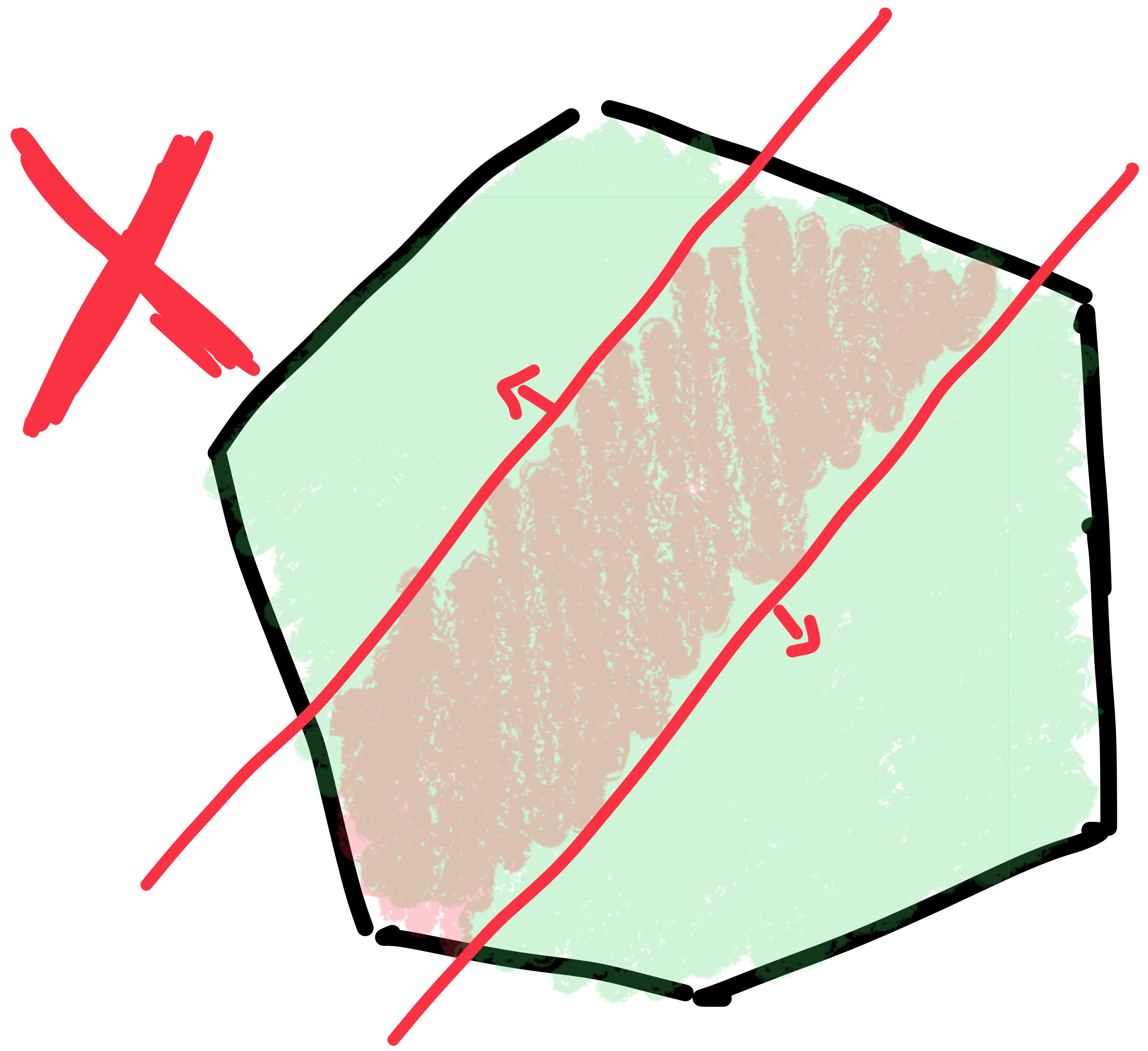

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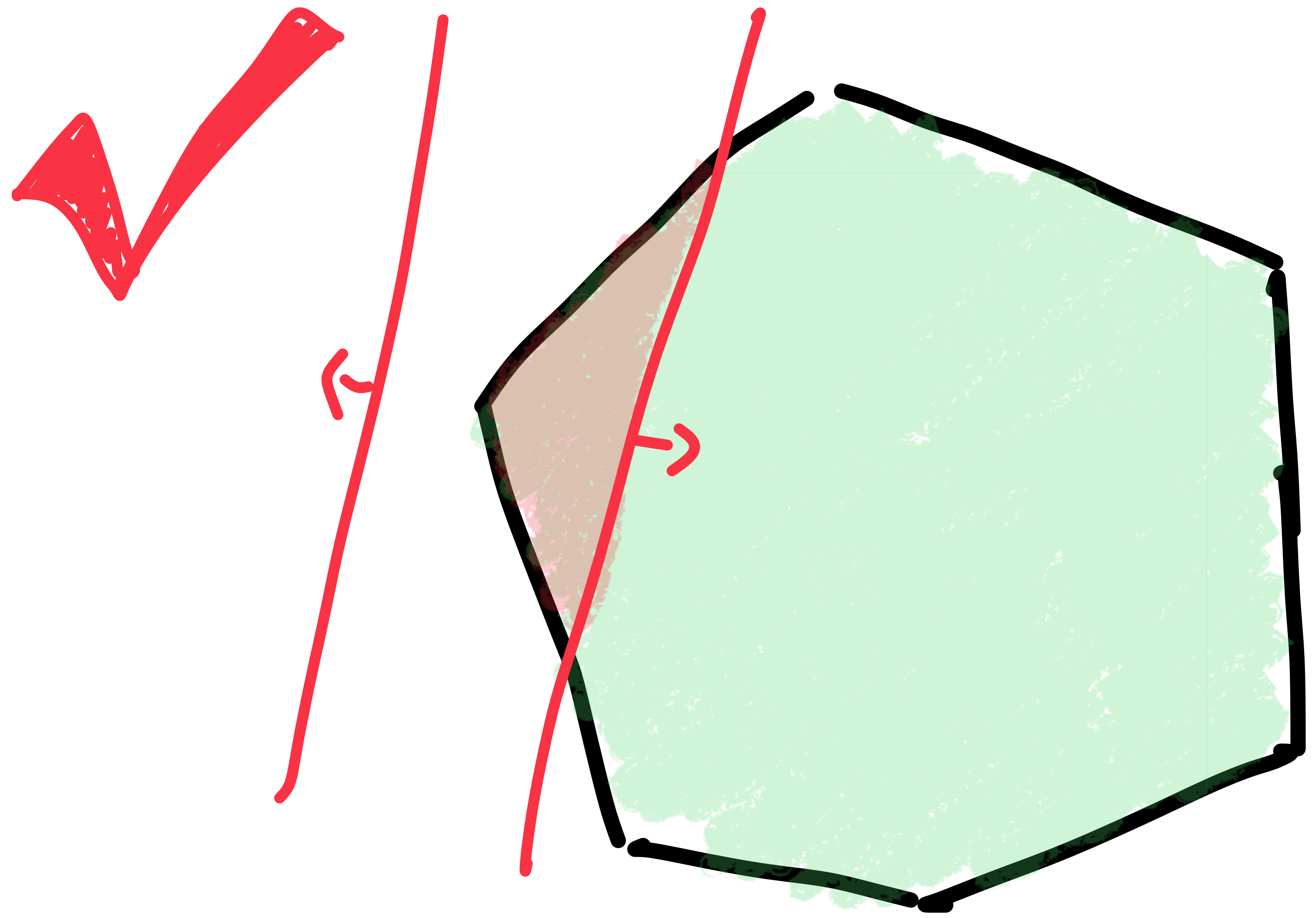

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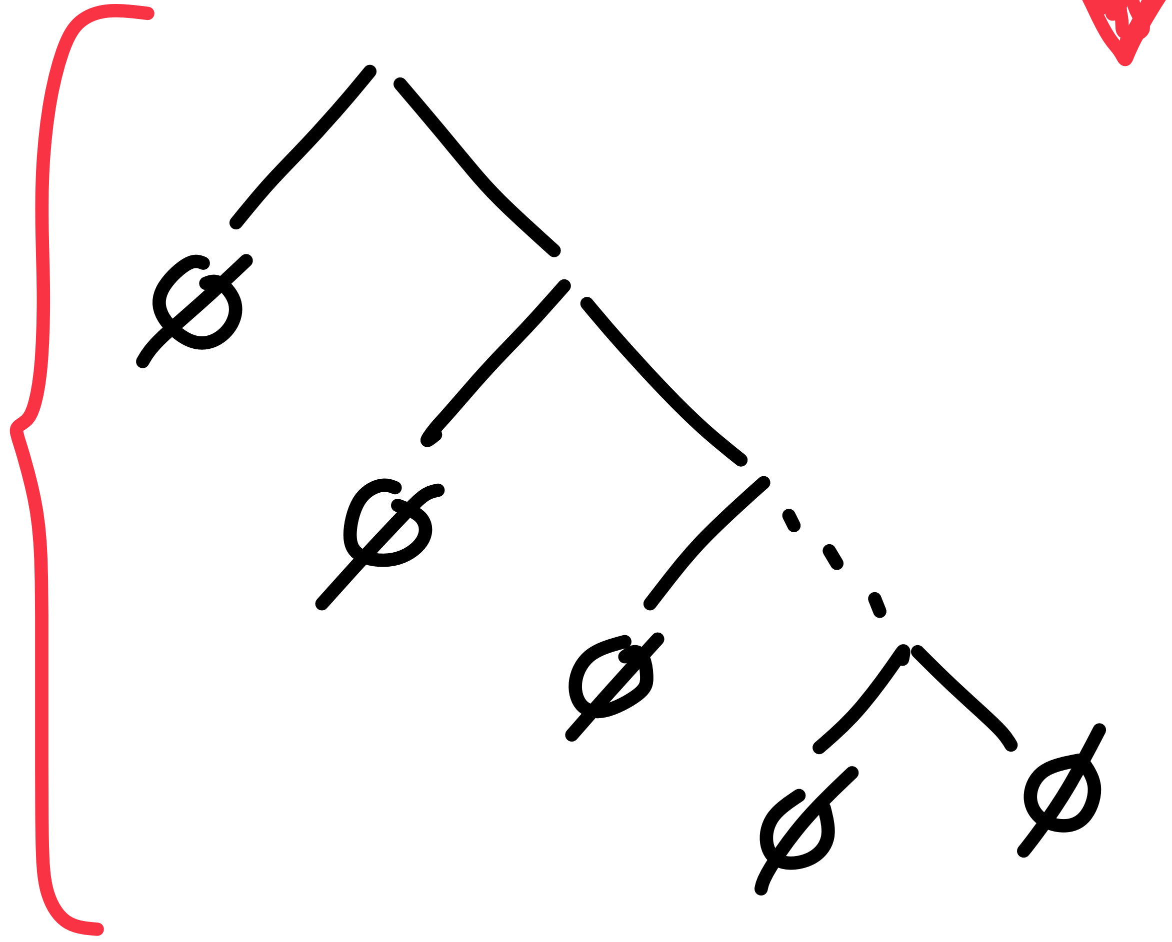
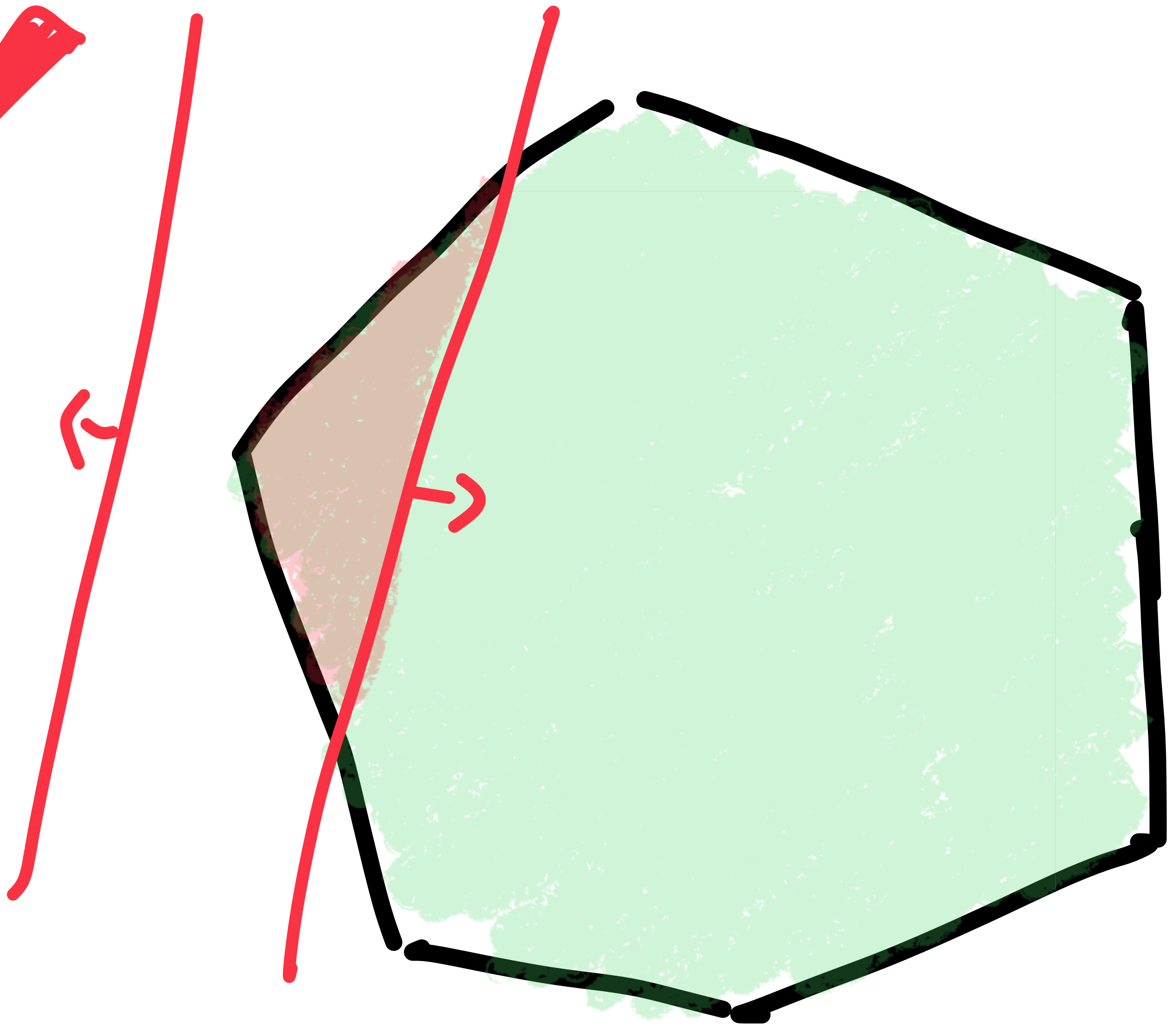
Pathlike SP Proof





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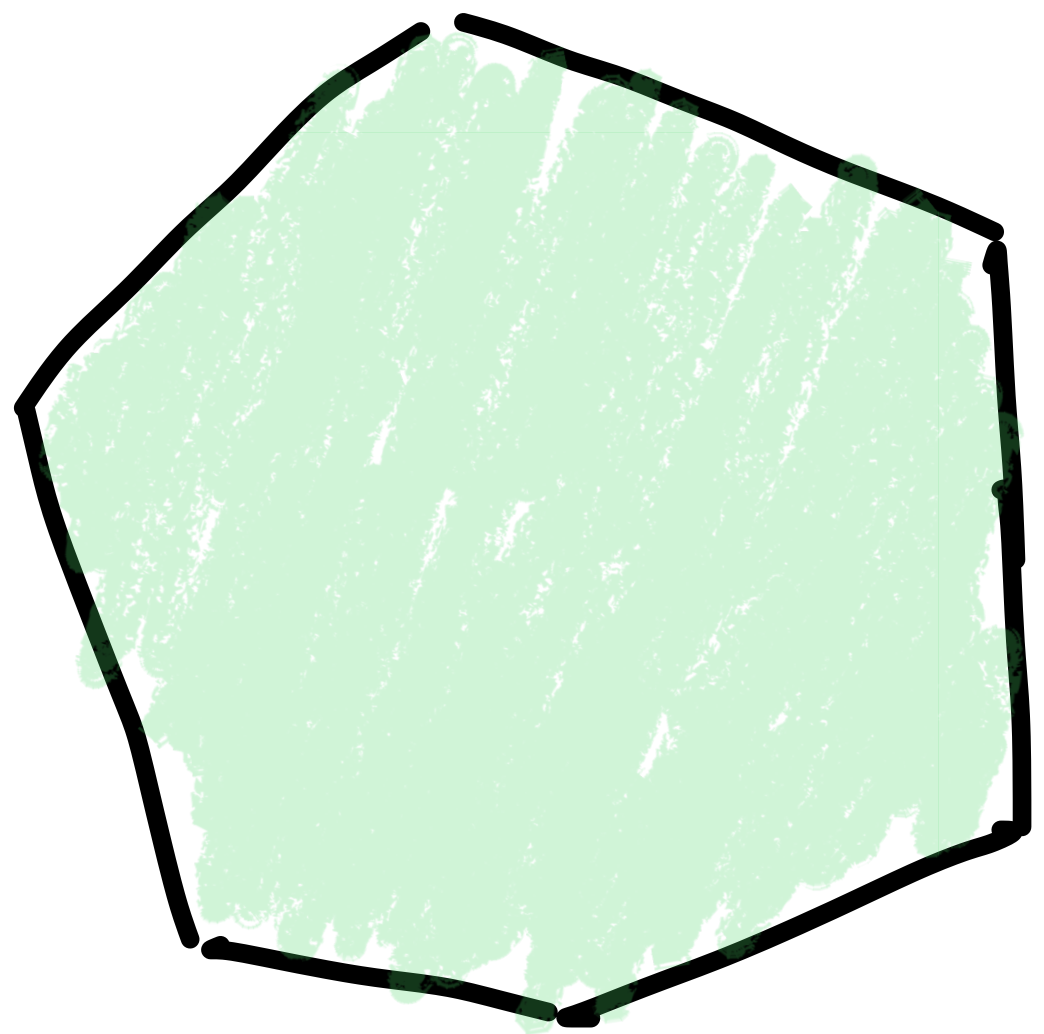

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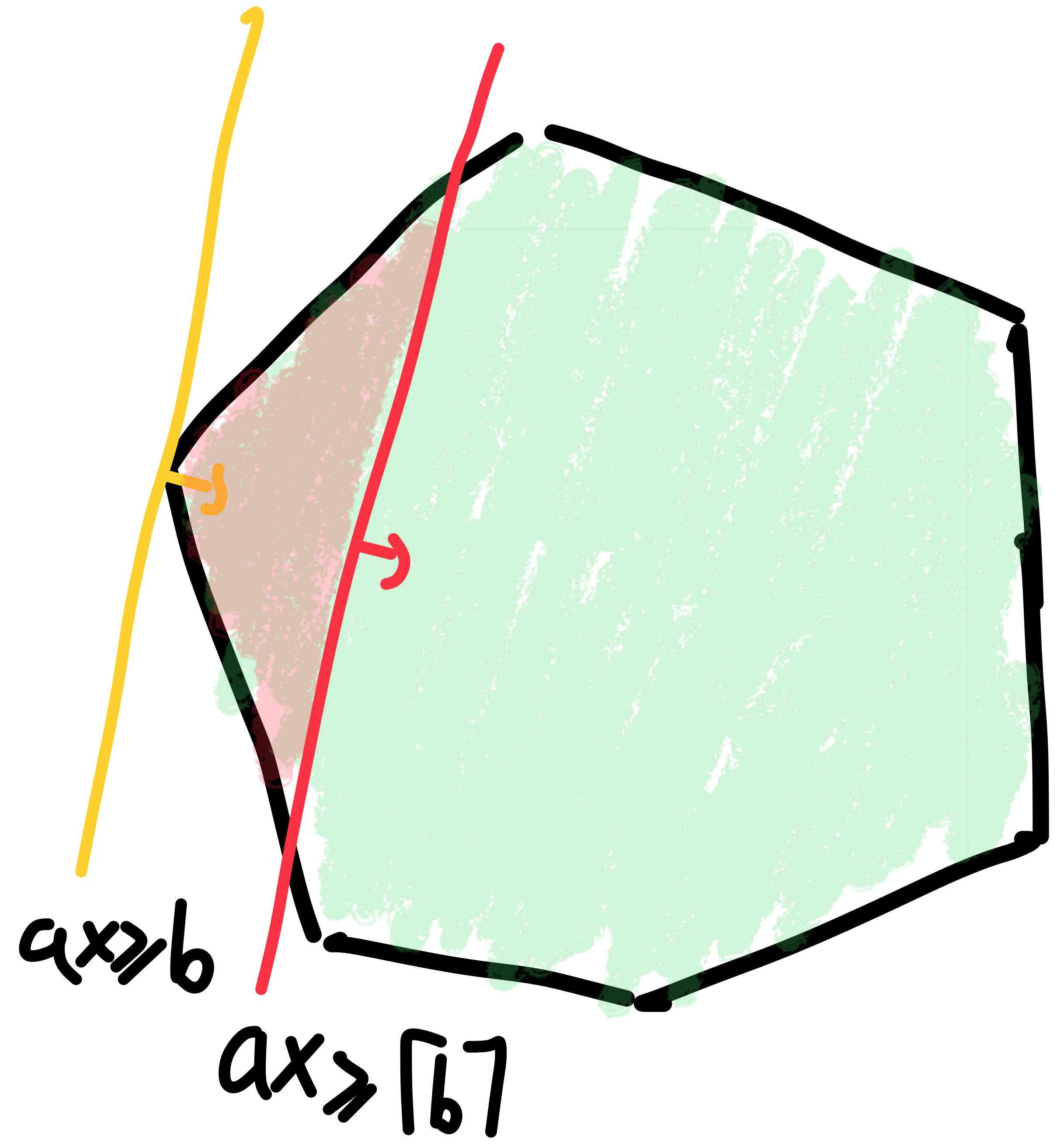
$ax \geq \lceil b \rceil$ is a CG-cut for P if $ax \geq b$ is valid for P



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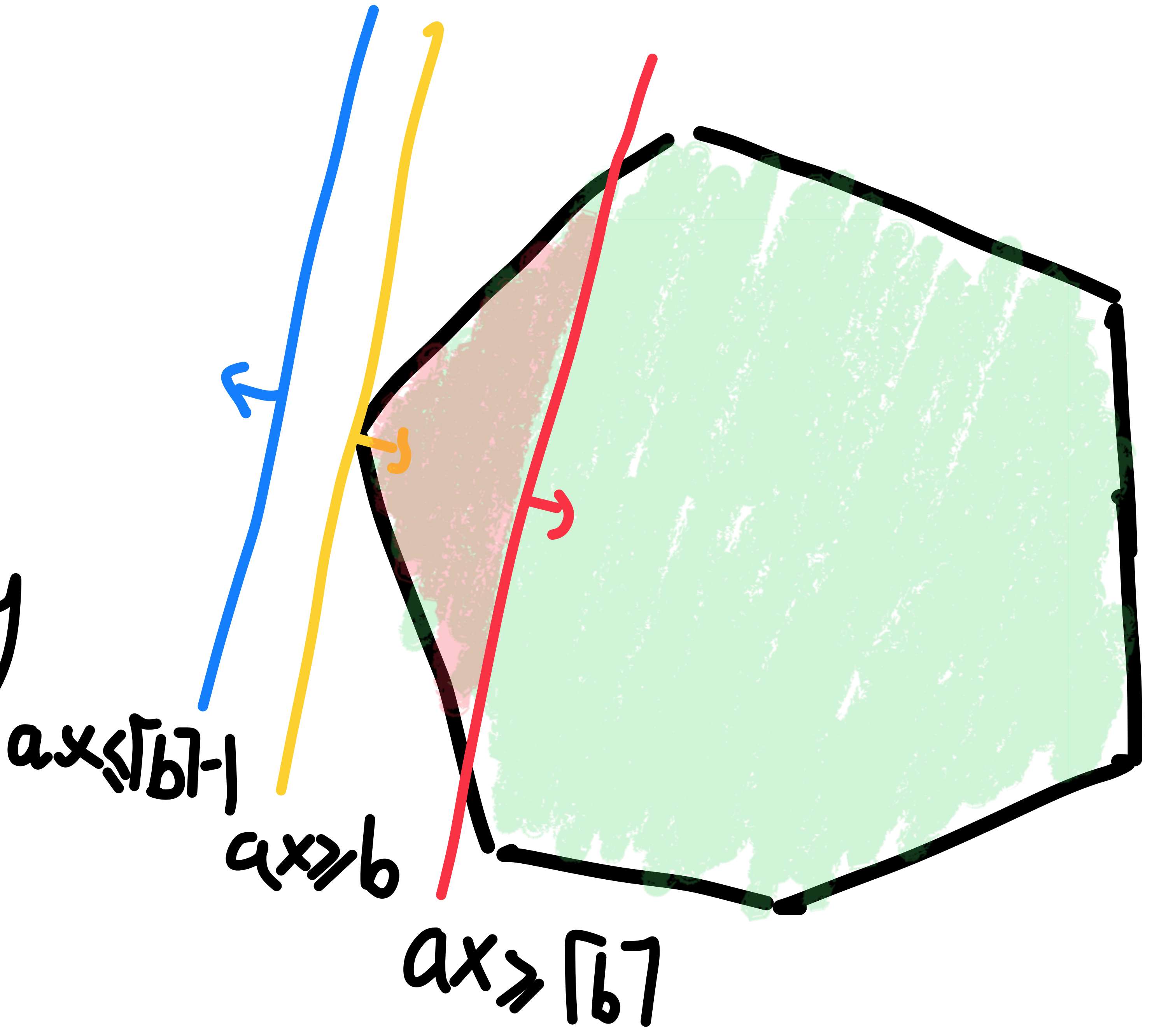
$\Rightarrow P \cap \{ax \leq \lceil b \rceil - 1\} = \emptyset$



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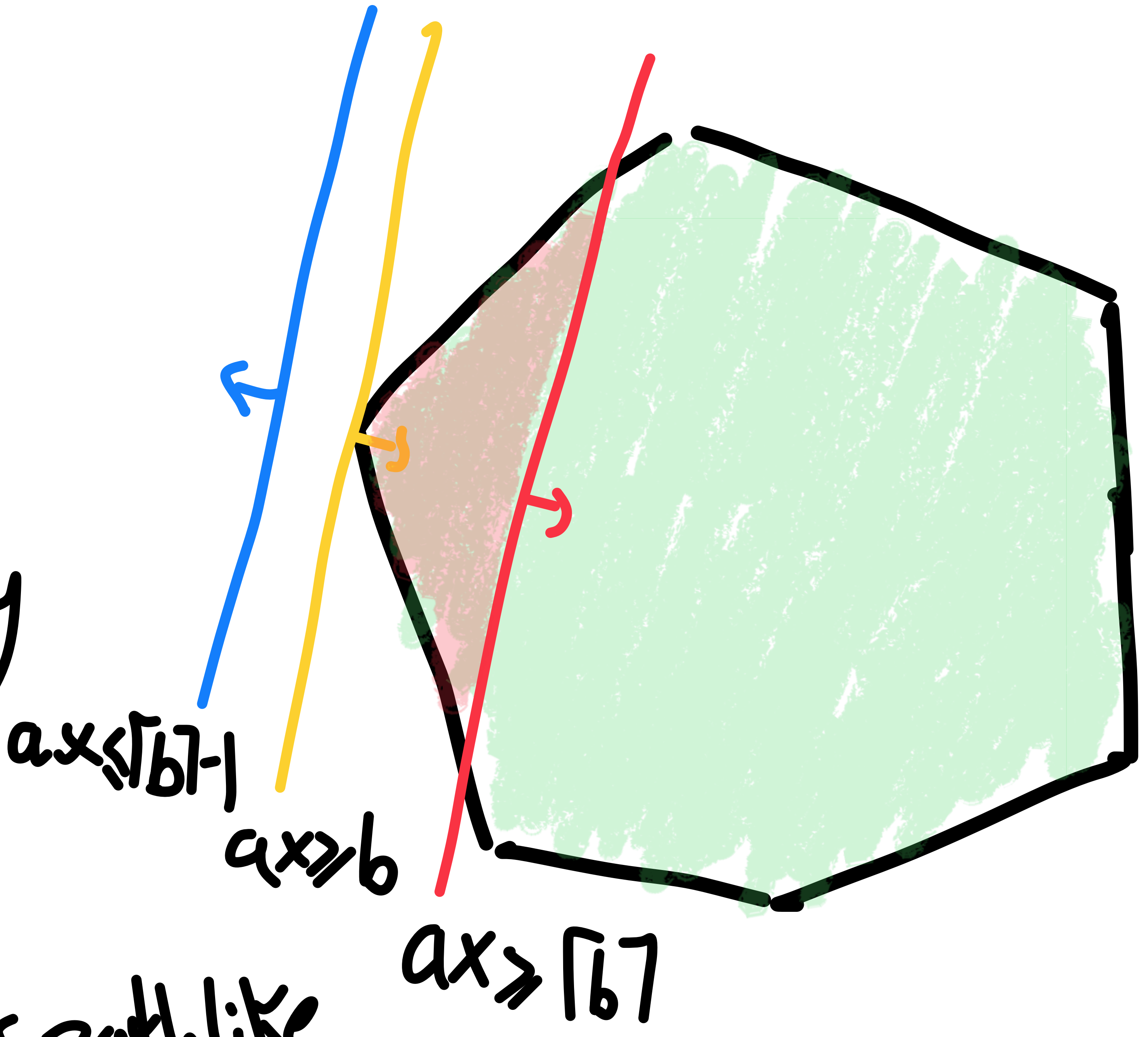

Stabbing Planes vs Cutting Planes

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[DT20] There are quasi-poly size CP proofs of Tseitin

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Stabbing Planes vs Cutting Planes

Thm: Every SP* proof can be quasipolynomially translated into CP

Stabbing Planes vs Cutting Planes

Thm: Every SP^* proof can be quasipolynomially translated into CP

Coefficients in queries are quasi-poly bounded

Stabbing Planes vs Cutting Planes

Thm: Every SP^* proof can be quasipolynomially translated into CP

→ SP^* is a "query" proof system (like DPLL) for CP

Stabbing Planes vs Cutting Planes

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Pf: a) CP = pathlike SP = facelike SP

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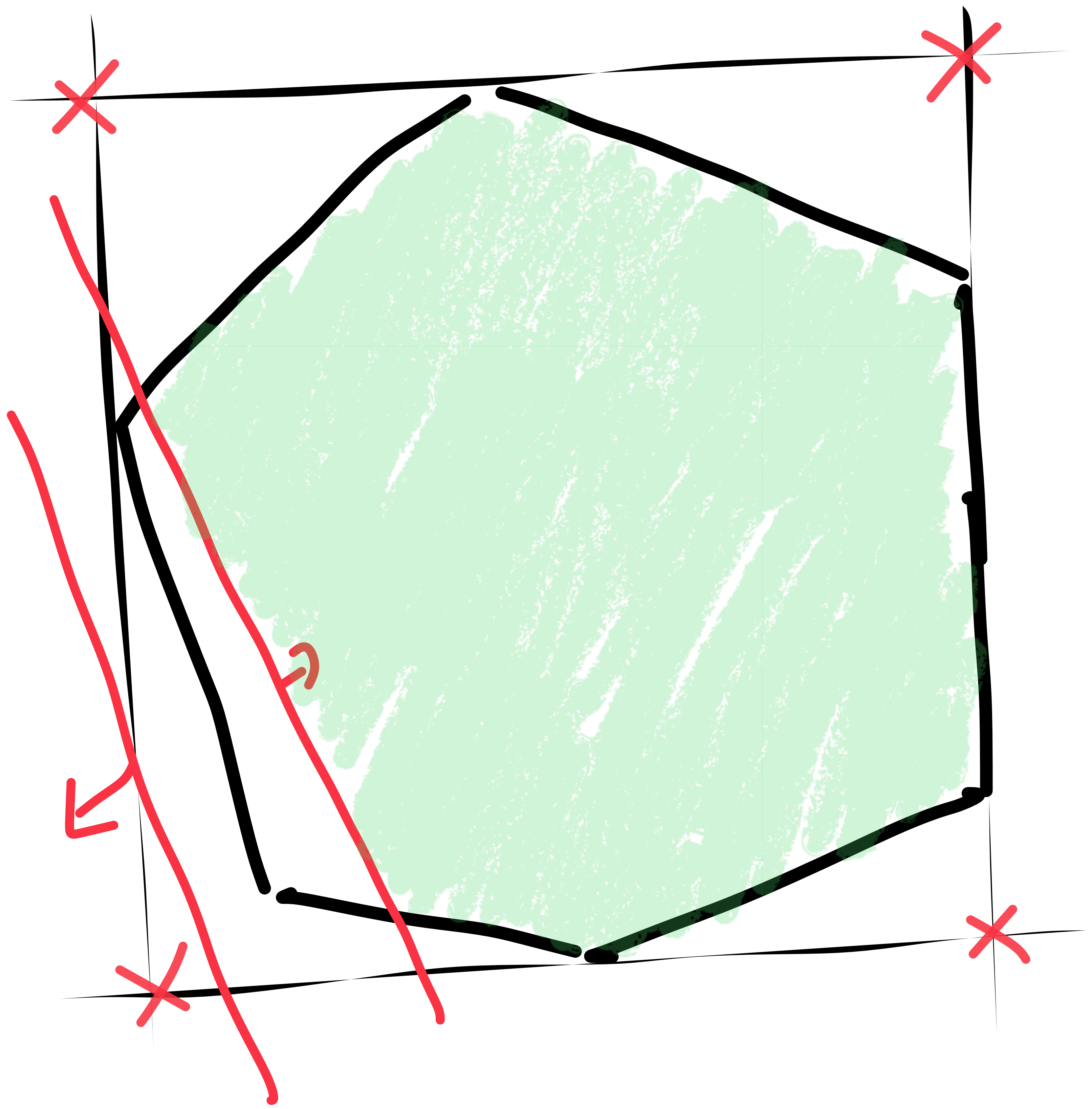

Stubbing Planes vs Cutting Planes

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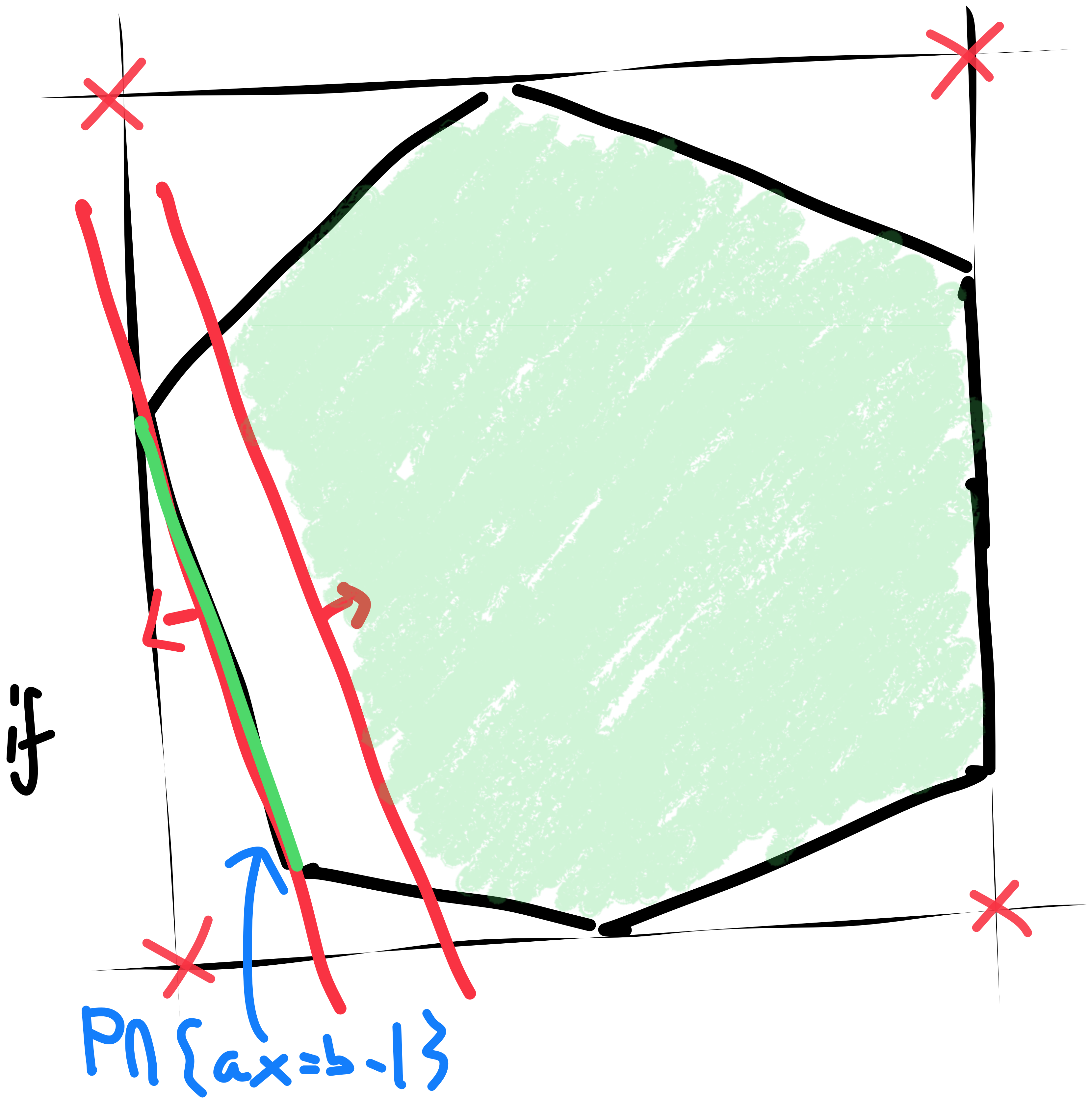

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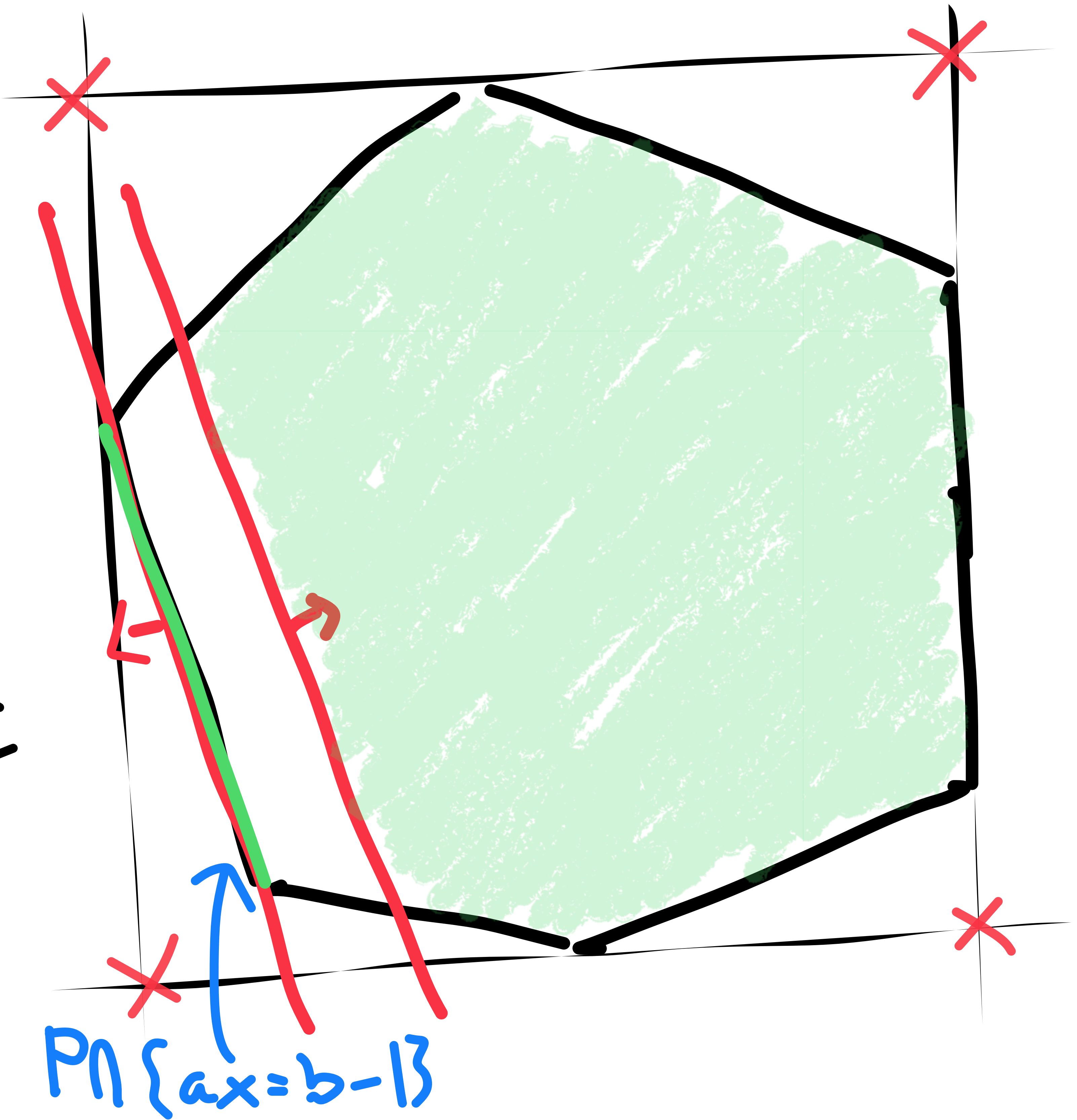
ie $P \cap \{ax \leq b-1\}$ or $P \cap \{ax \geq b\}$ is a face.



Stabbing Planes vs Cutting Planes

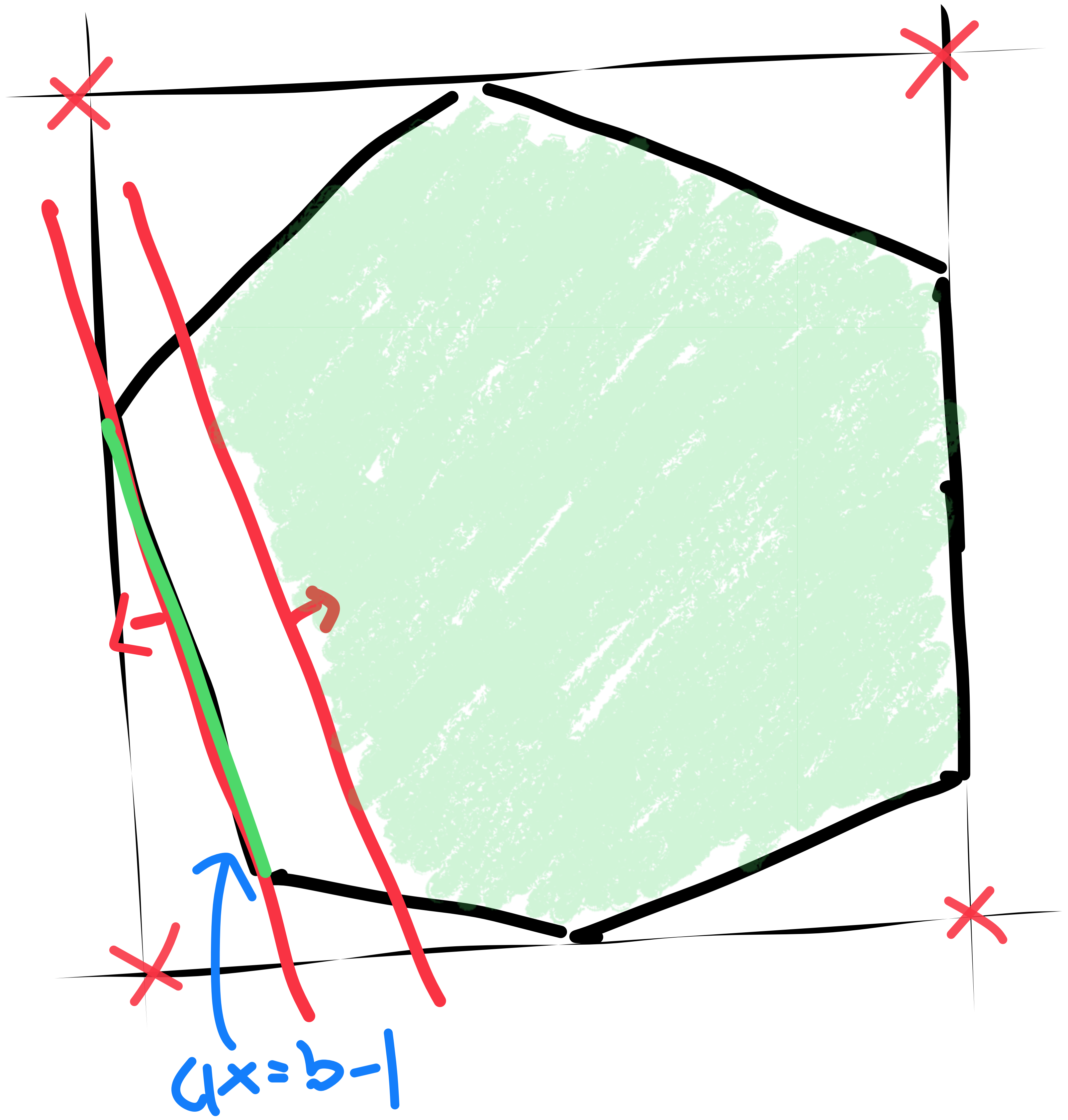
Thm: Every SP^* proof can be quasipolynomially translated into CP

Pf: a) CP = pathlike SP = facelike SP

b) Any SP^* proof can be made facelike with a quasi-poly blowup in size.

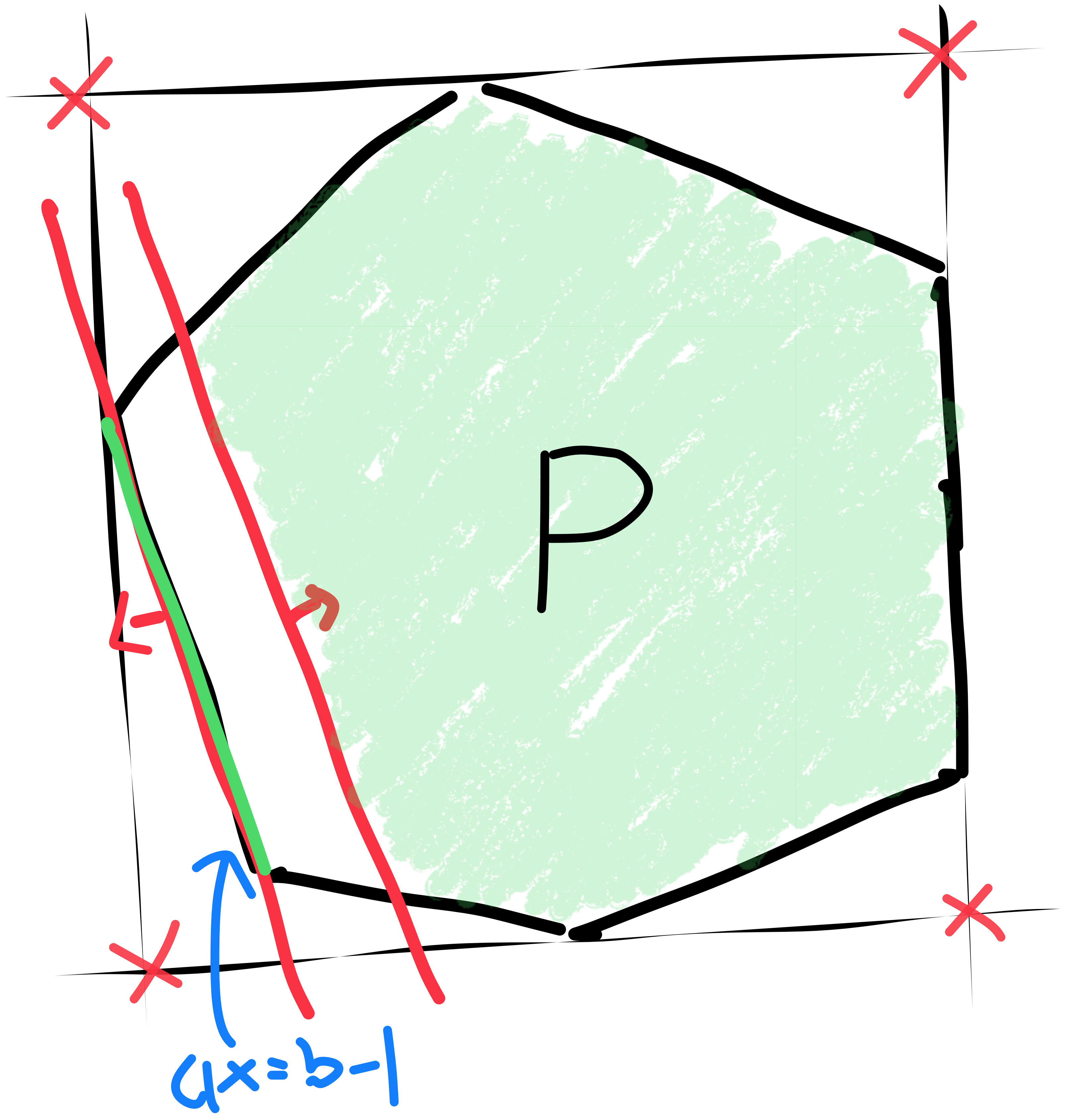
CP = Facelike SP

e.g.



CP = Facelike SP

Observation: If we can refute $P \cap \{ax = b-1\}$ then we can deduce $P \cap \{ax \geq b\}$



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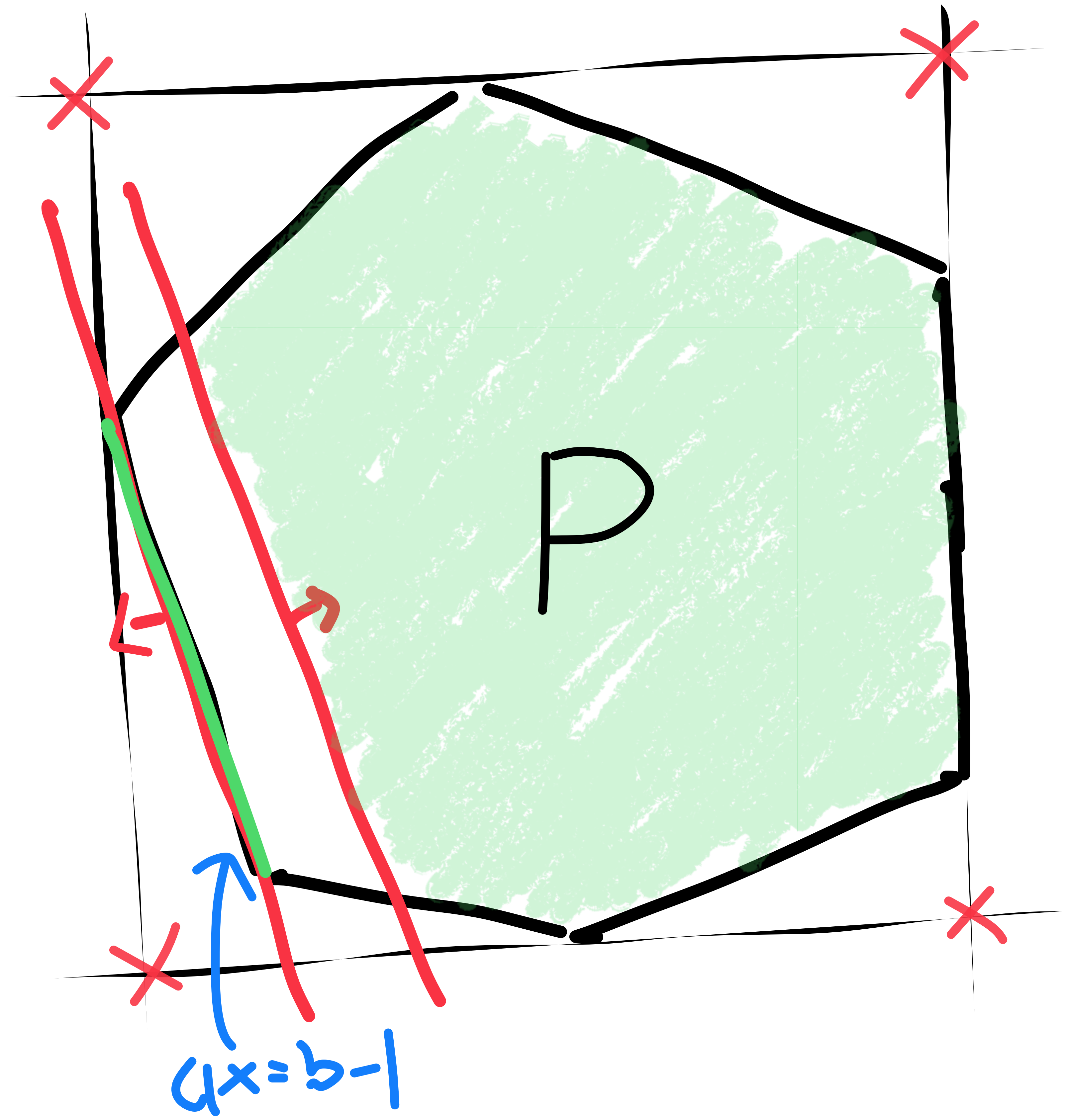

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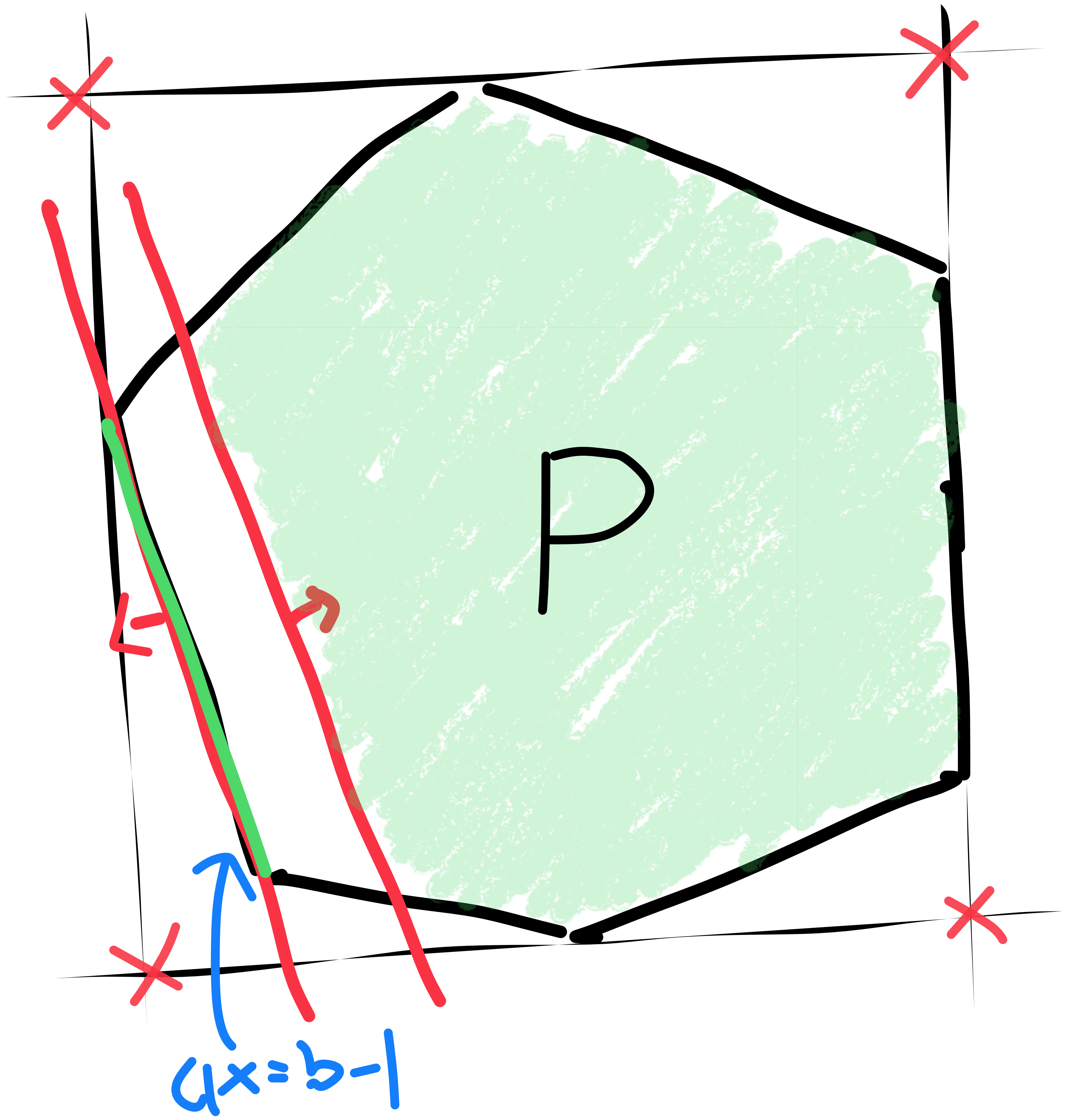
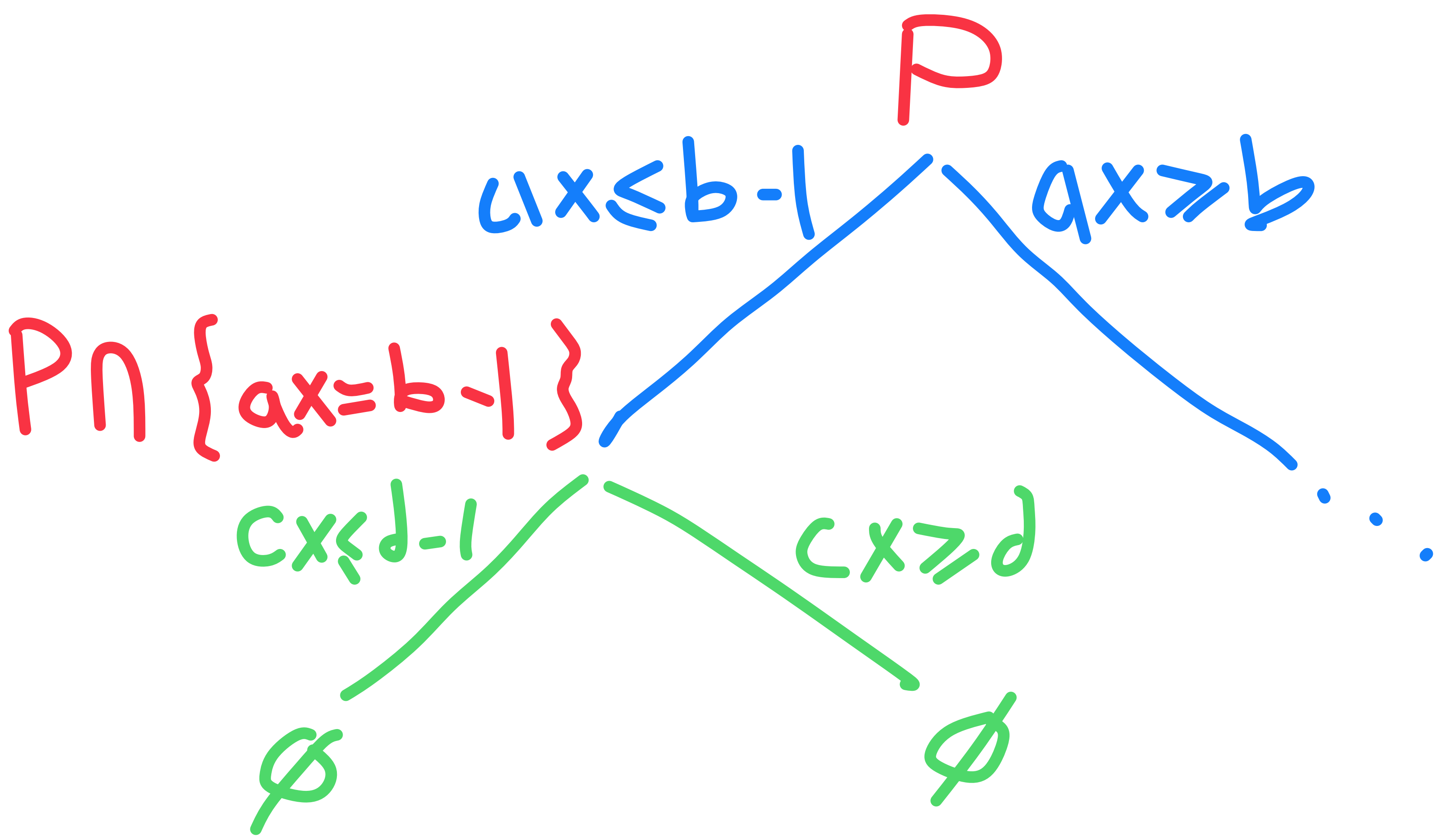
Consider the partial Facelike SP proof





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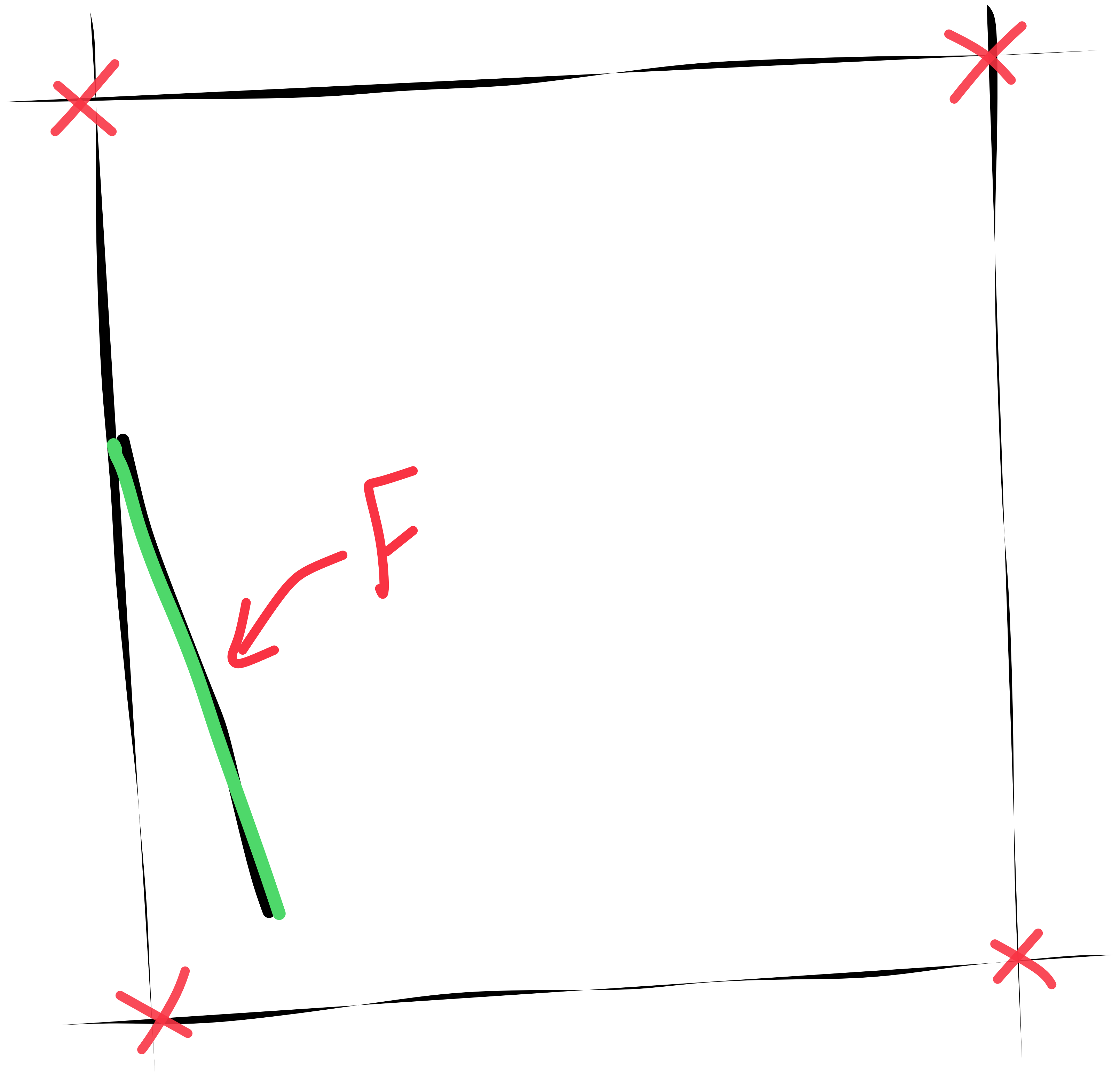

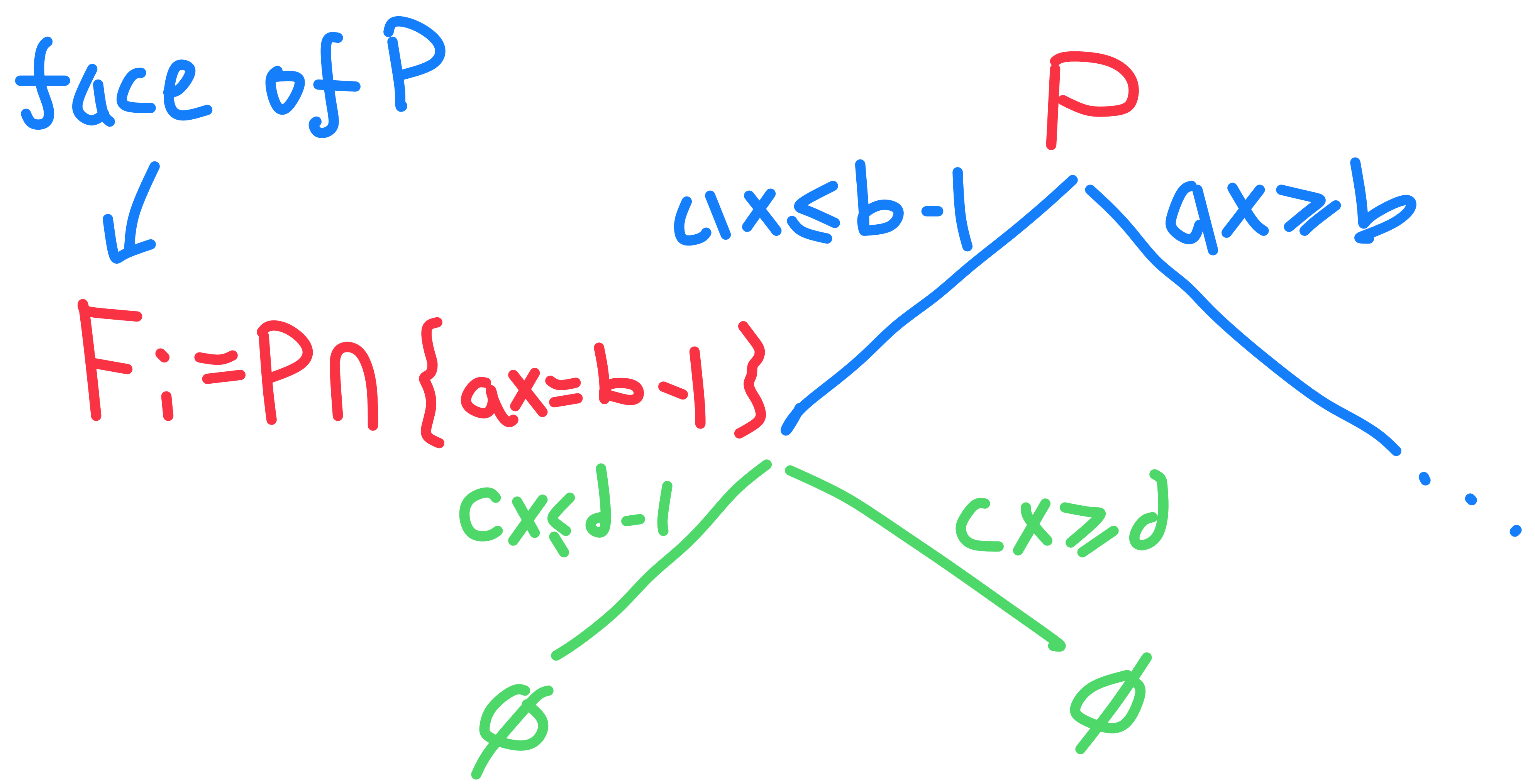

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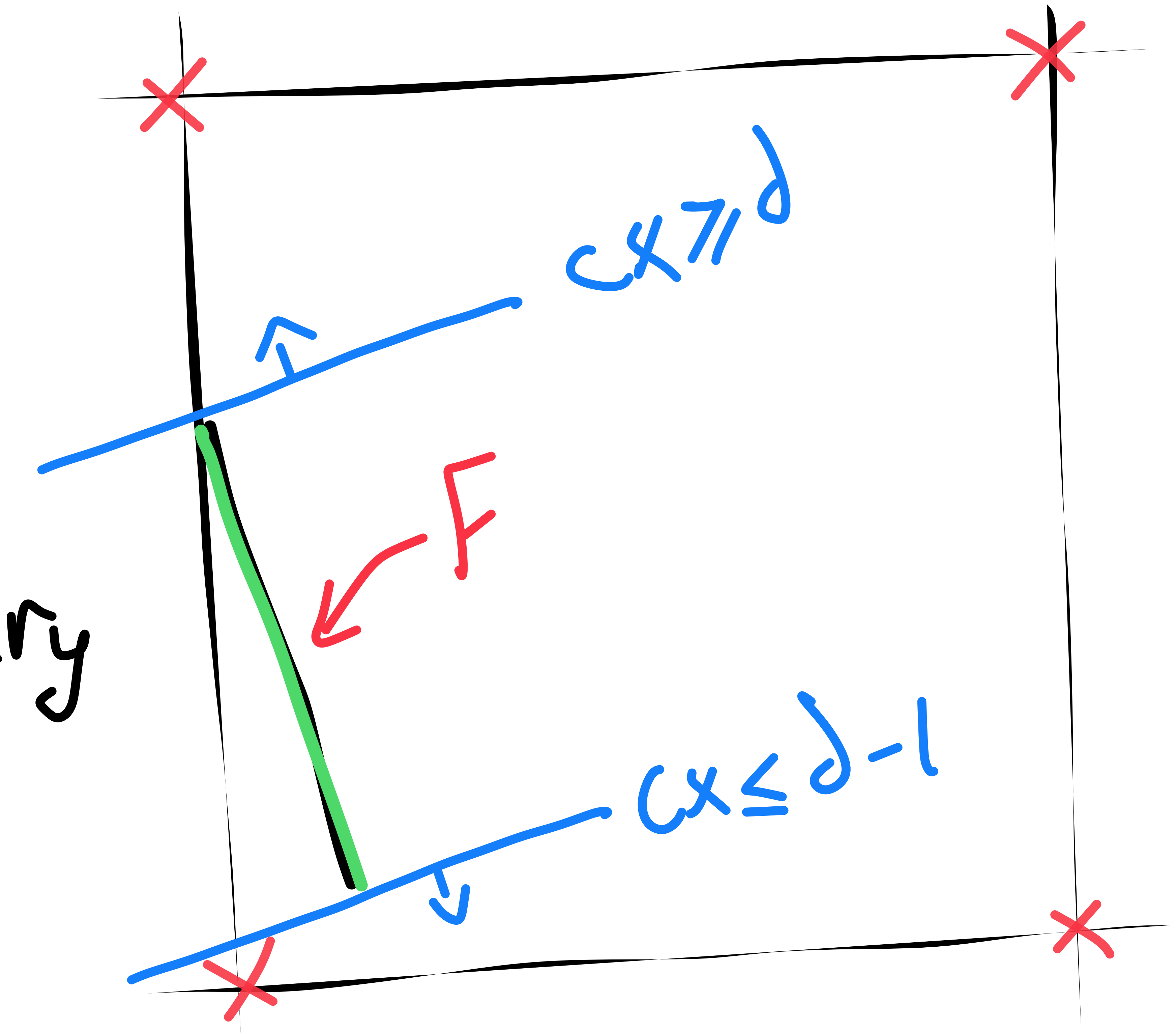

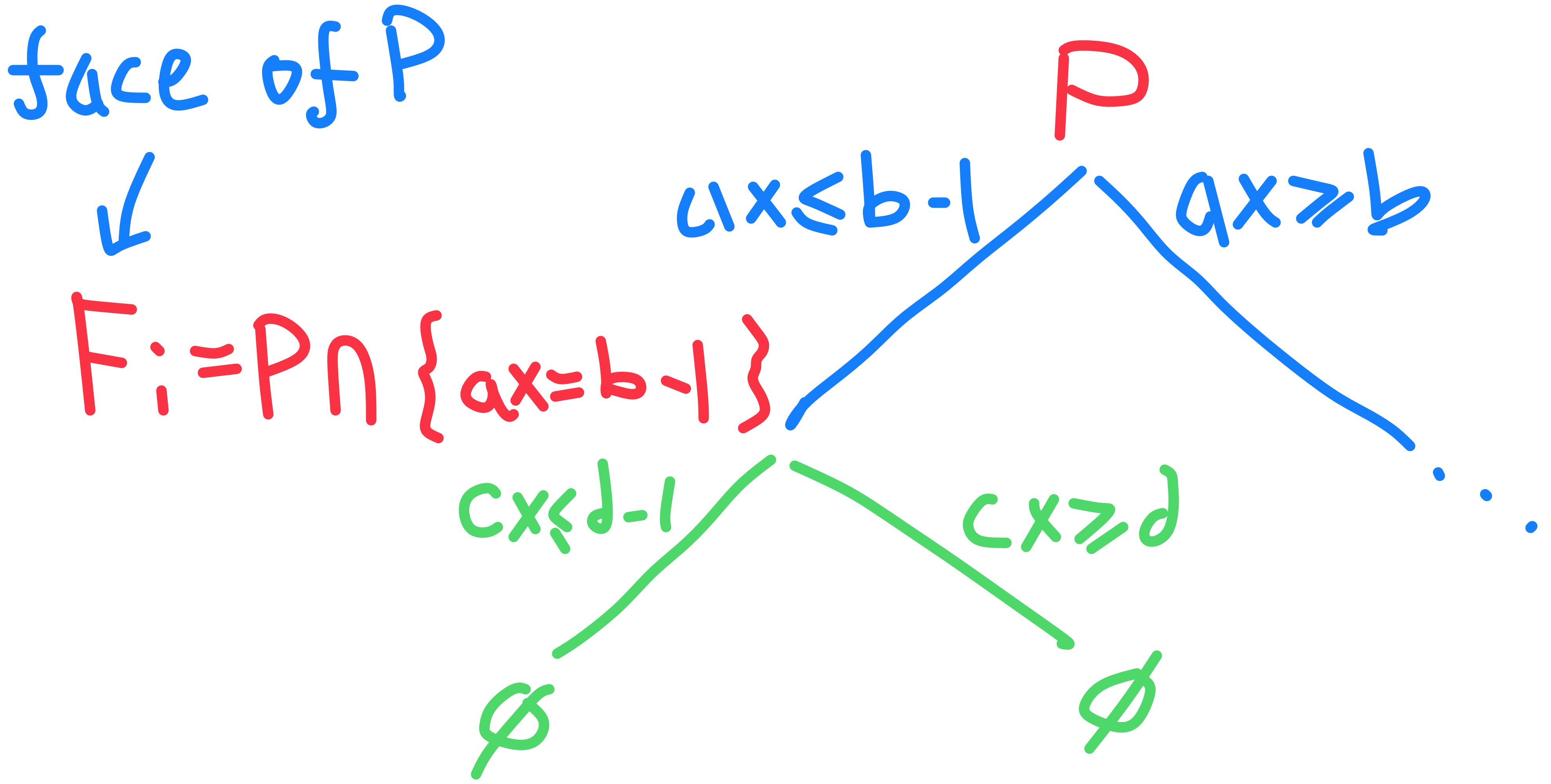

CP = Facelike SP

Observation: If we can refute $P \cap \{ax = b-1\}$ then we can deduce $P \cap \{ax \geq b\}$

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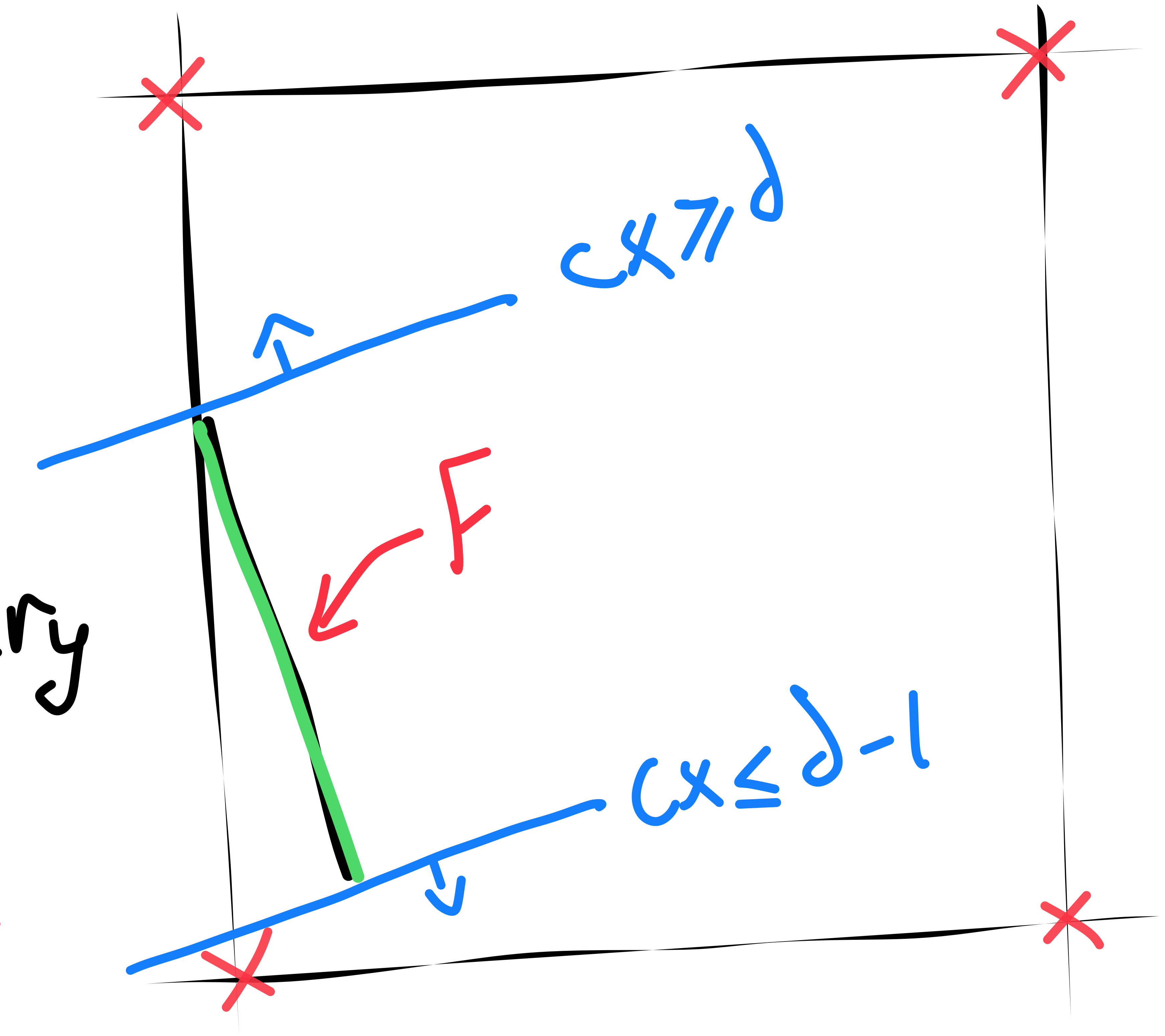
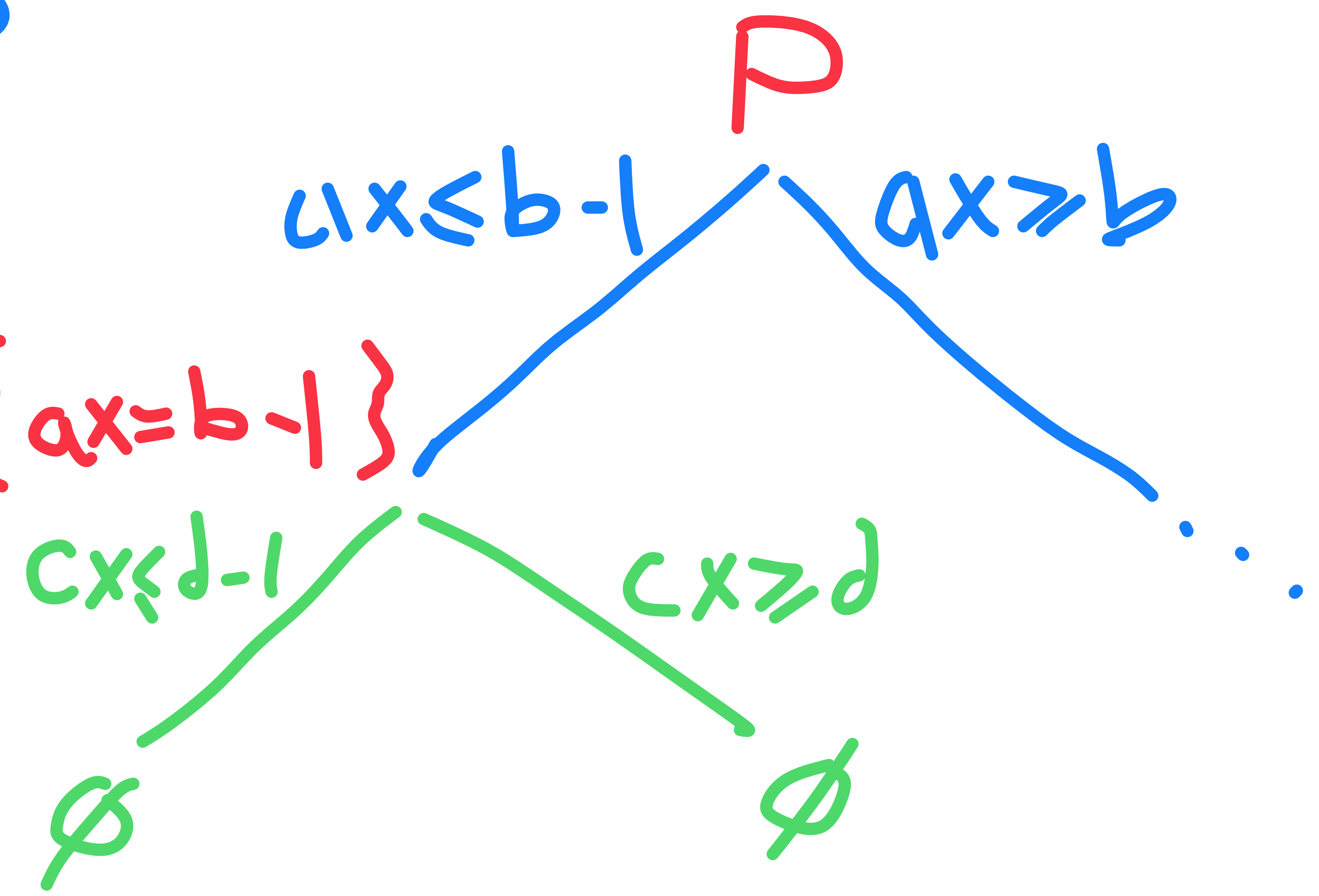
Observe: $(cx \leq d-1, cx \geq d)$ is a pathlike query for F



face of P

$F := P \cap \{ax = b-1\}$



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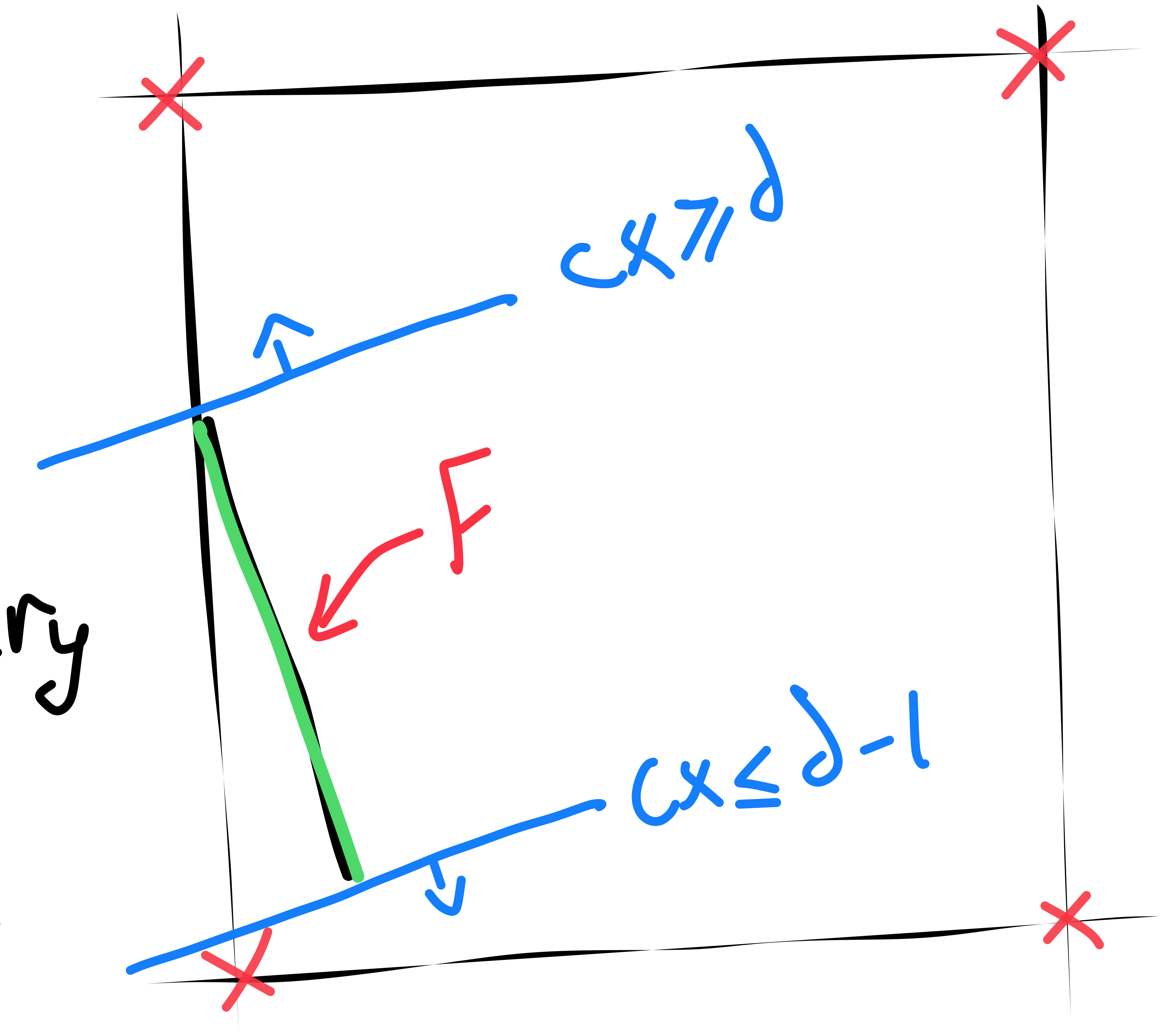

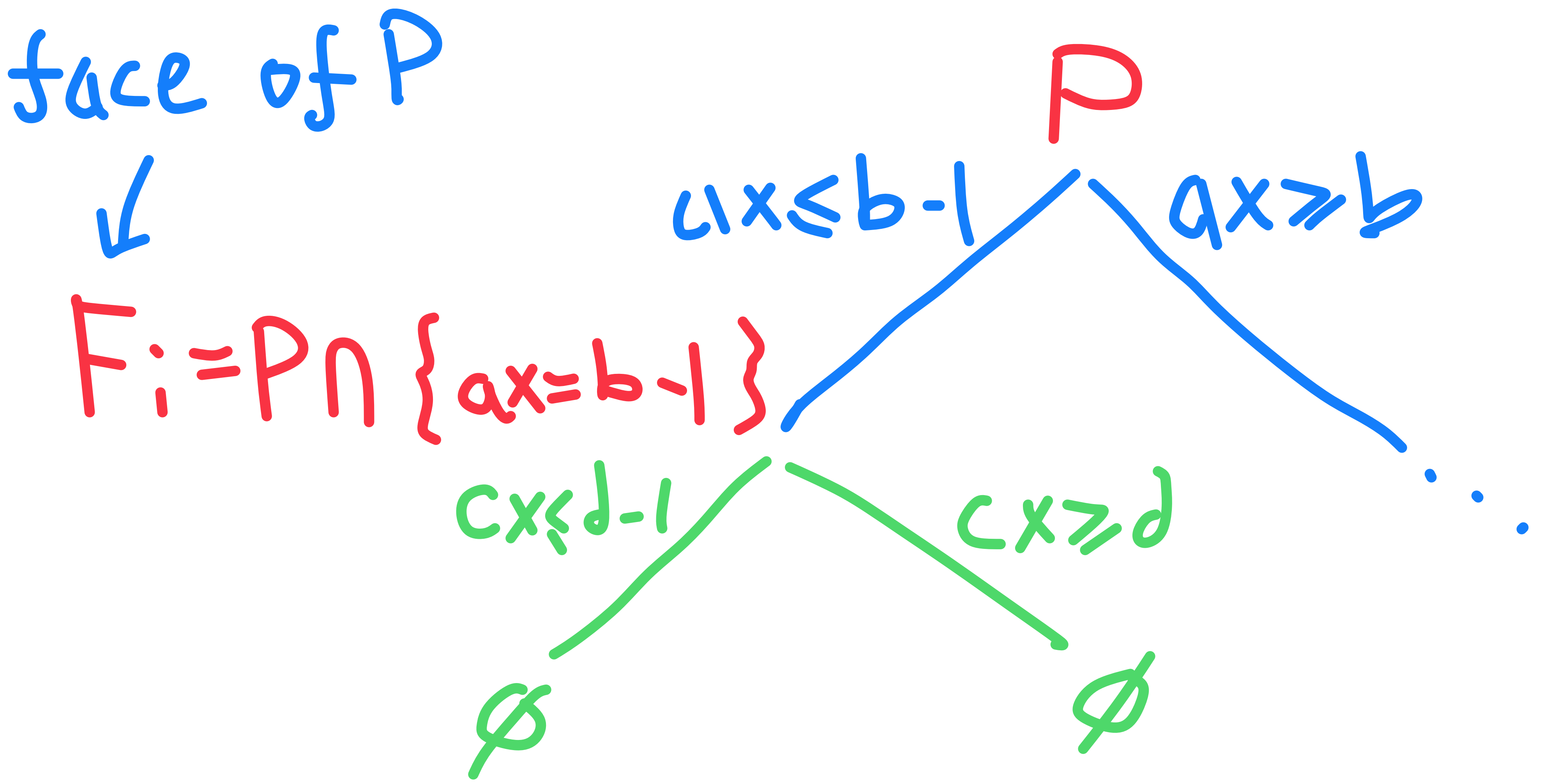

Q: Can we "simulate" the refutation of F when starting from P?
i.e. derive P' st $P' \cap F = \emptyset$?

CP = Facelike SP

Notation. Say $P \vdash P'$ if P' is obtained from P by a pathlike query

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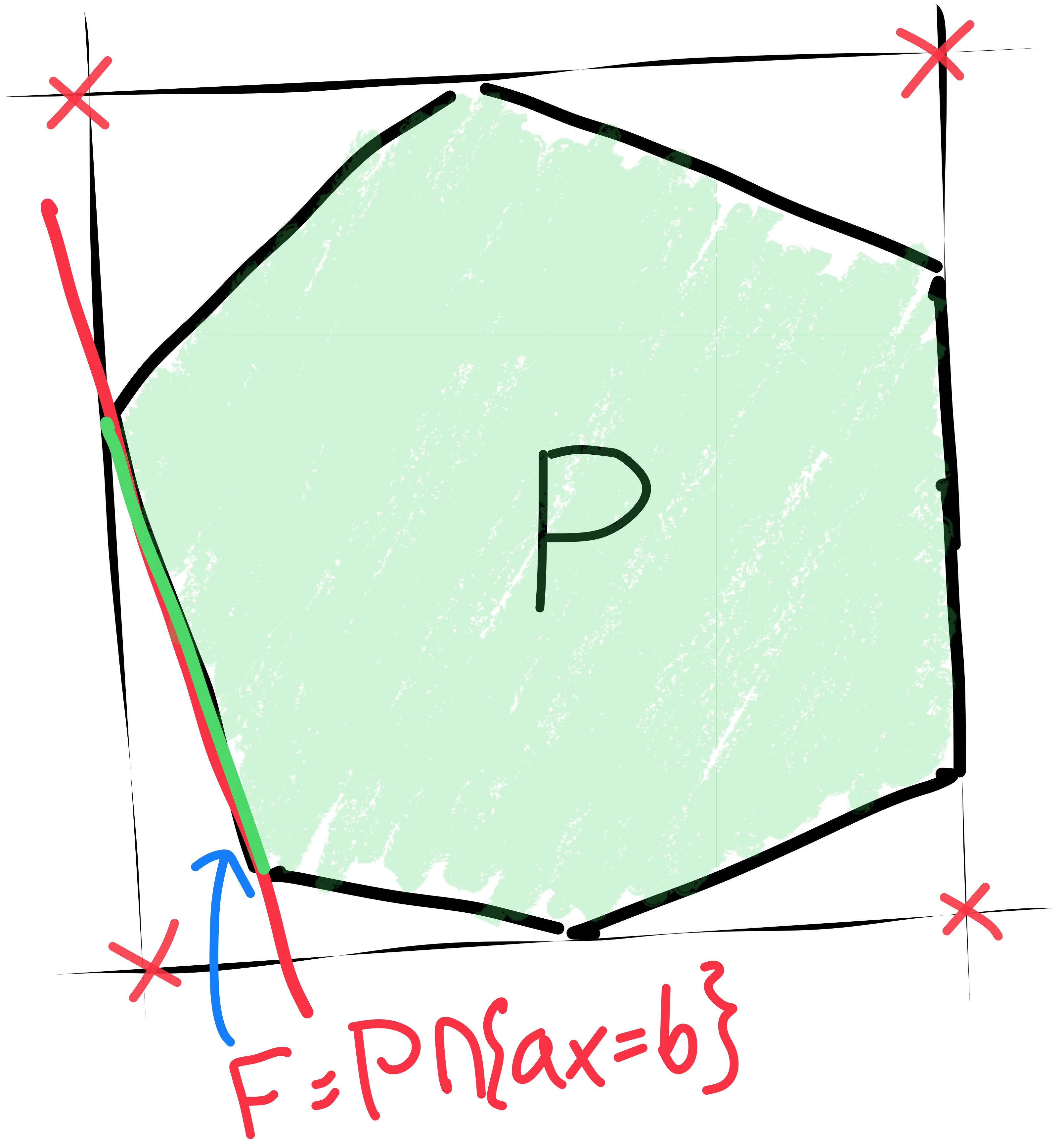

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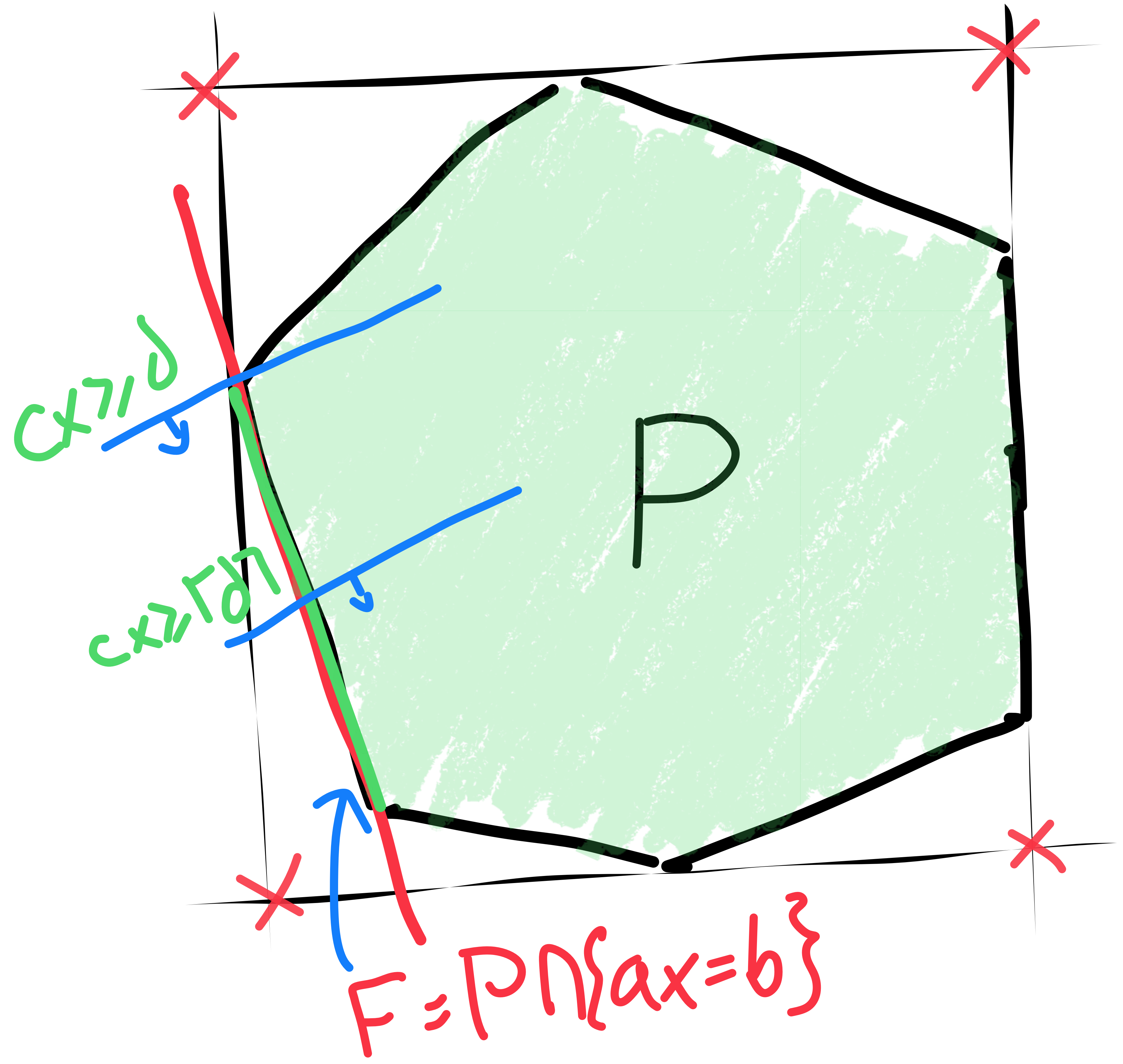

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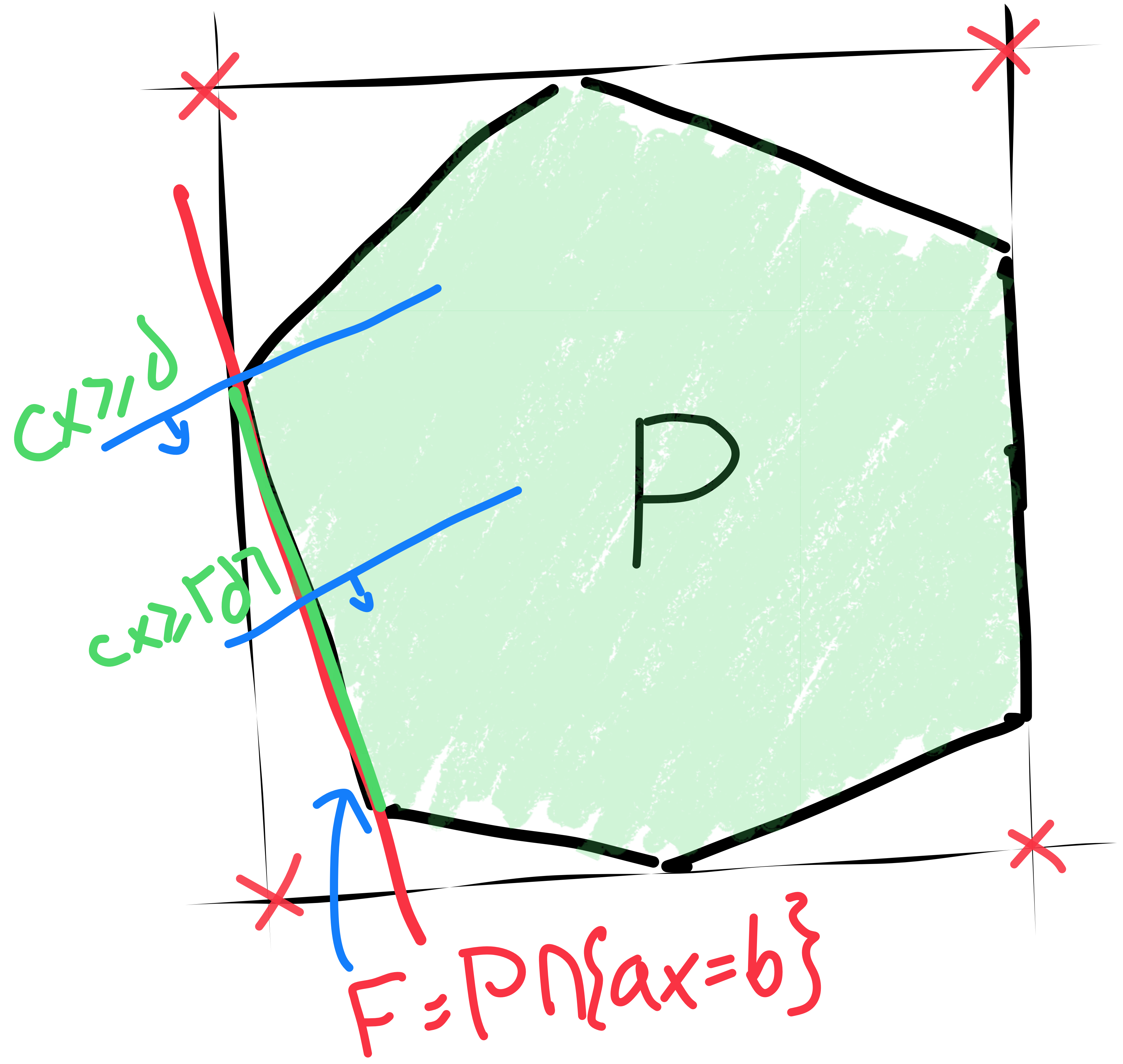
- $ax \leq b$ only inequality of F not valid for P



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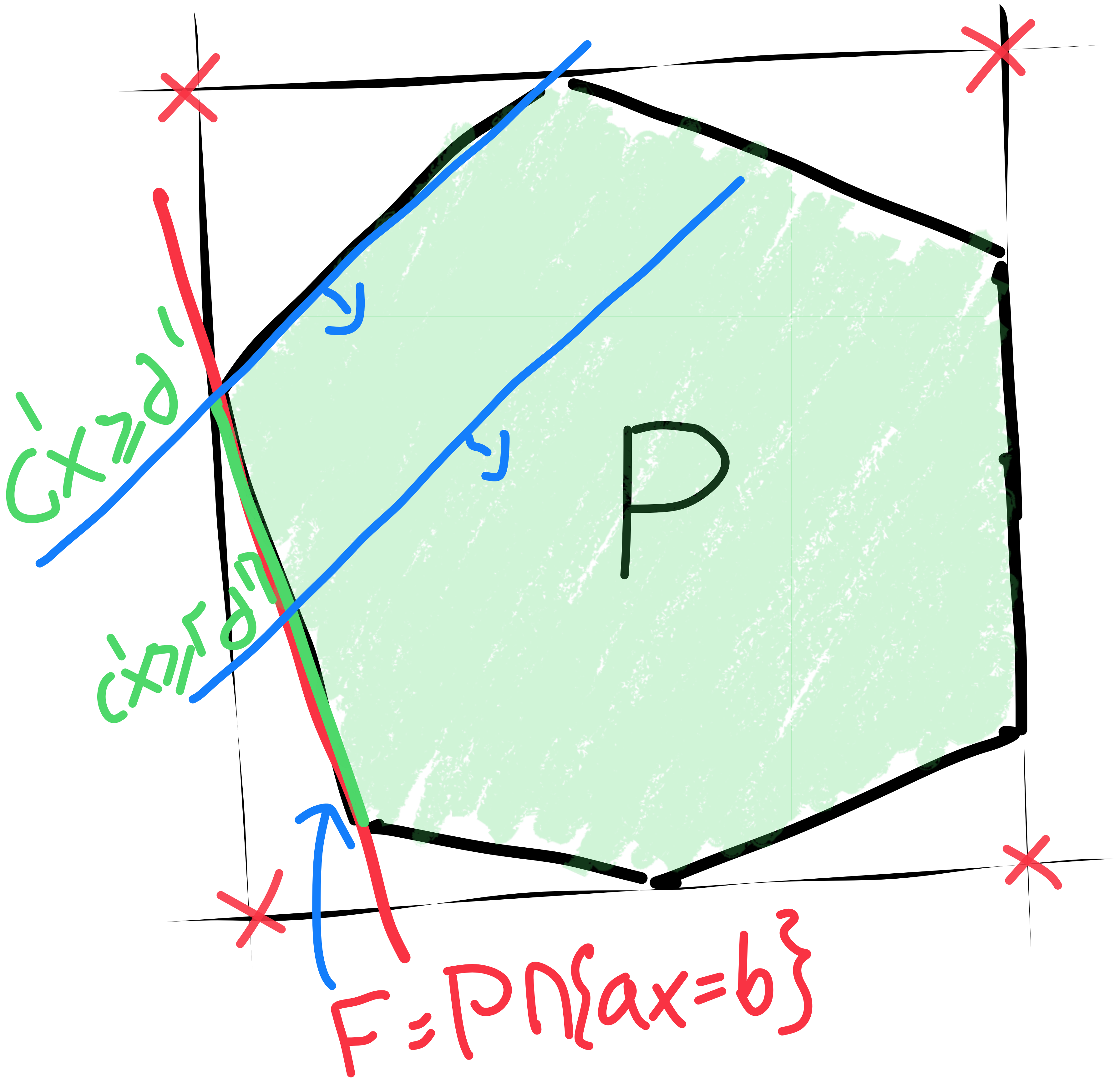

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$\therefore$ Shift $cx \geq d$ by $ax \geq b$ to $c'x \geq d'$ so it doesn't depend on $ax \leq b$

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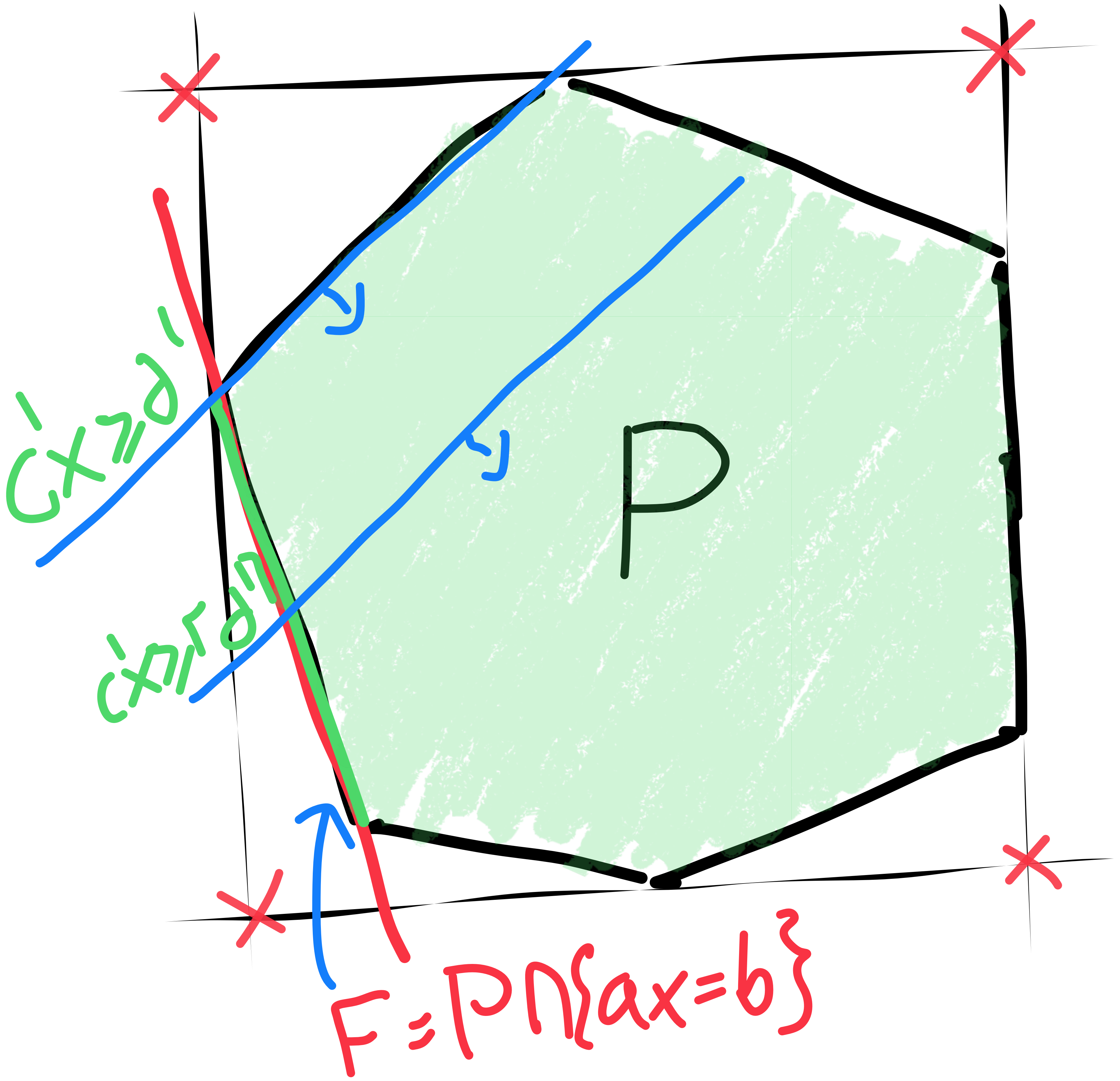
- $ax \leq b$ only inequality of F not valid for P
- ∴ Shift $cx \geq d$ by $ax \geq b$ to $c'x \geq d'$ so it doesn't depend on $ax \leq b$

$\Rightarrow P \cap \{c'x \geq \lceil d' \rceil\} \cap F \subseteq F'$



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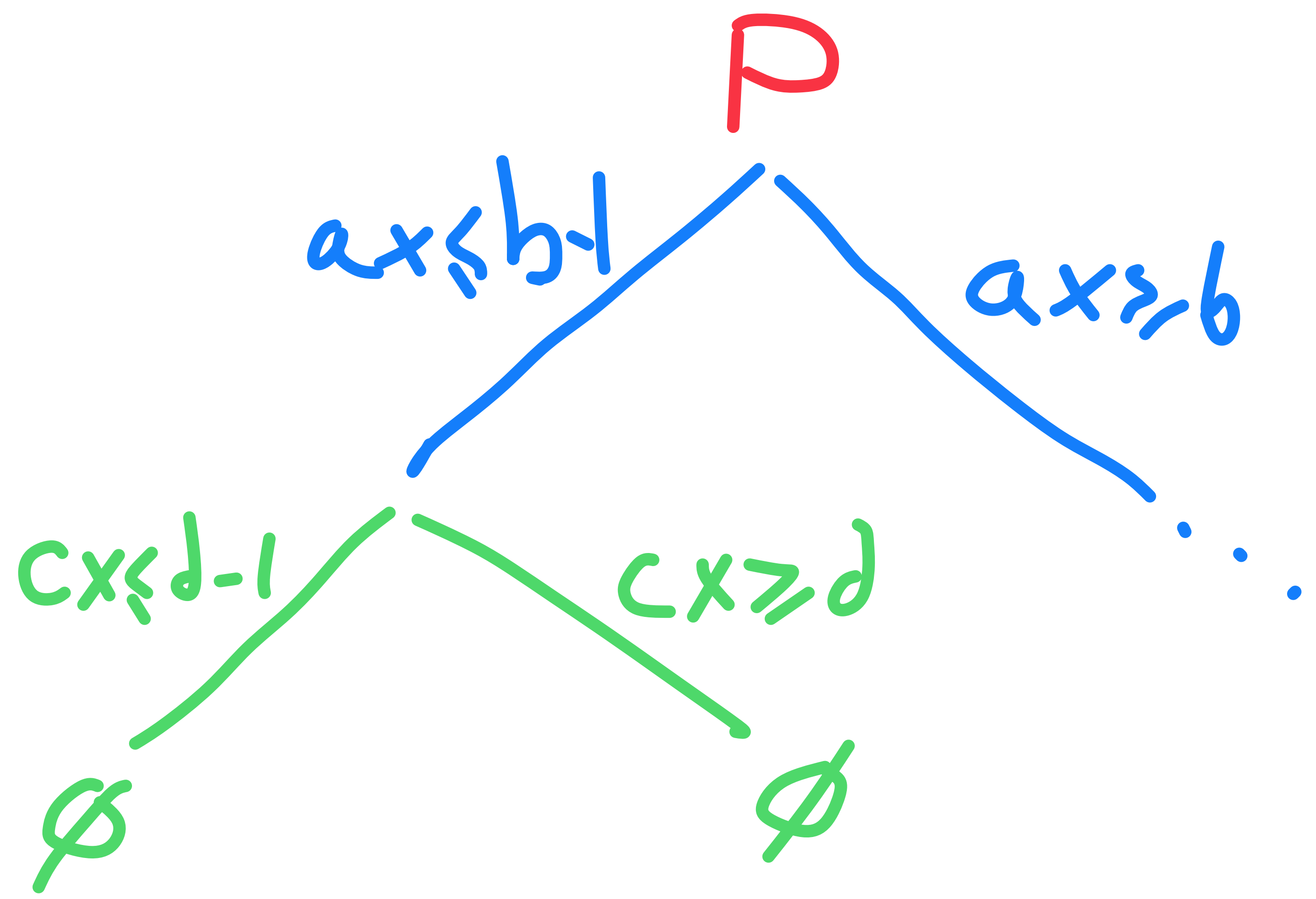

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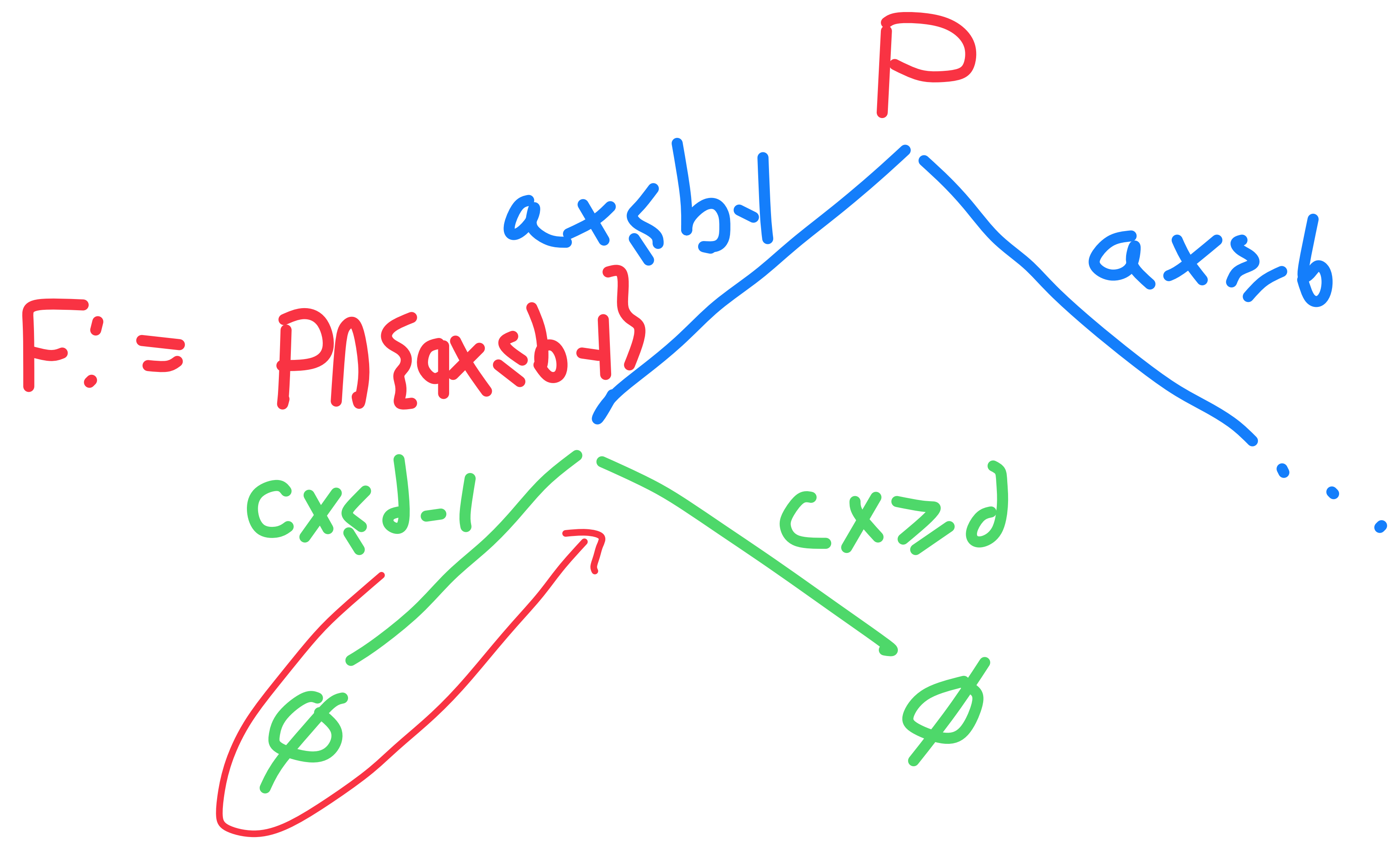

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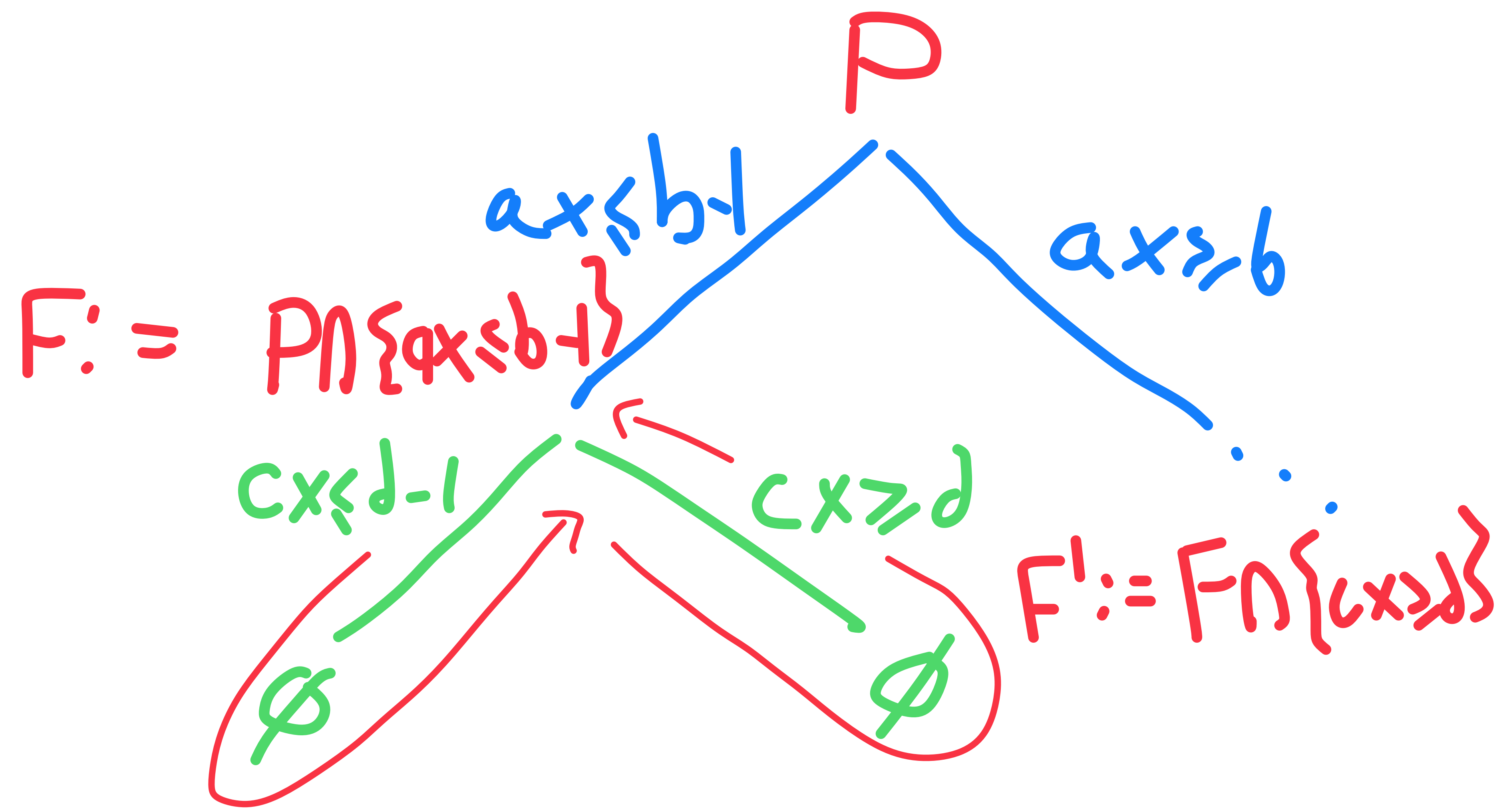

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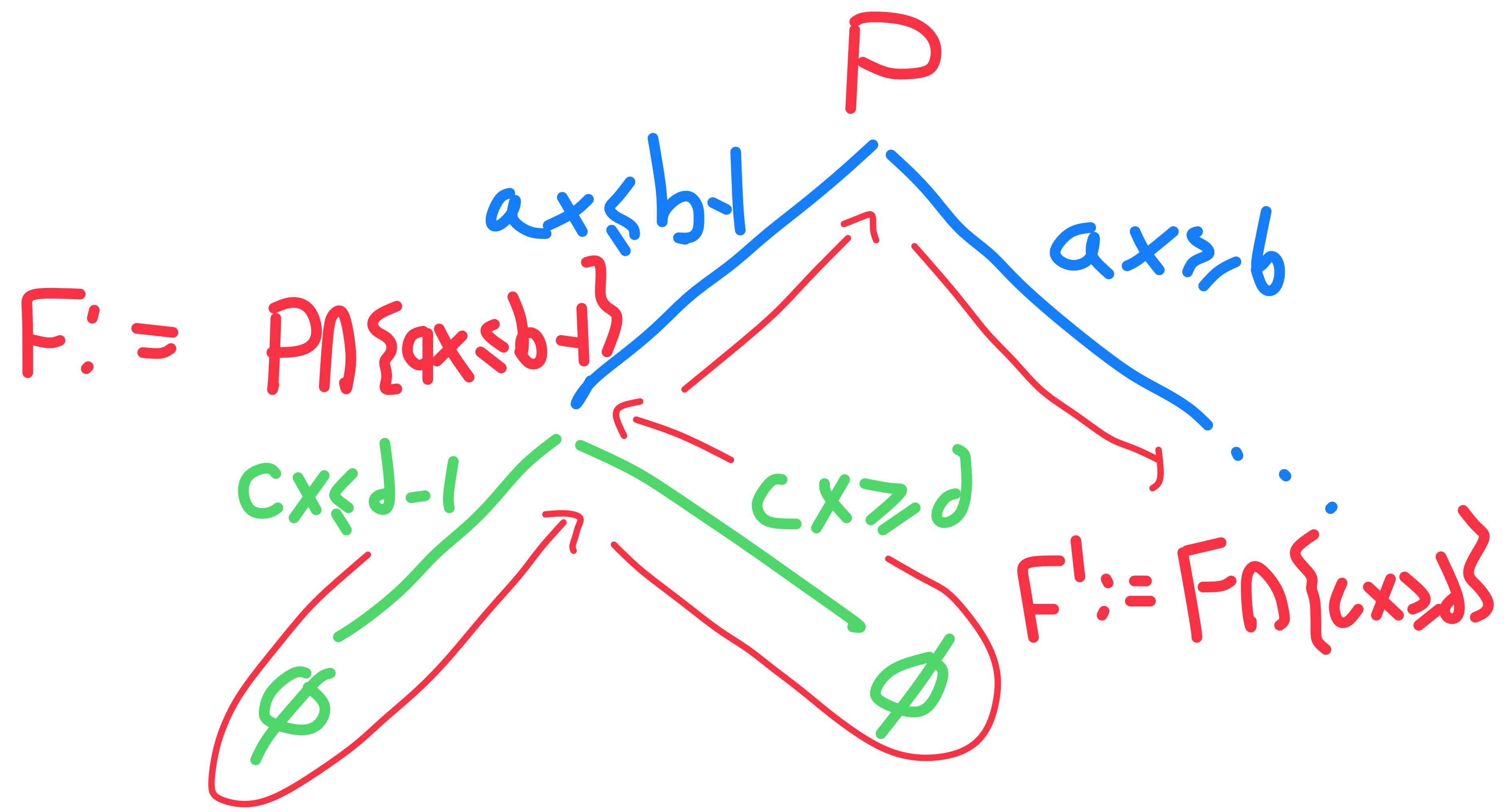

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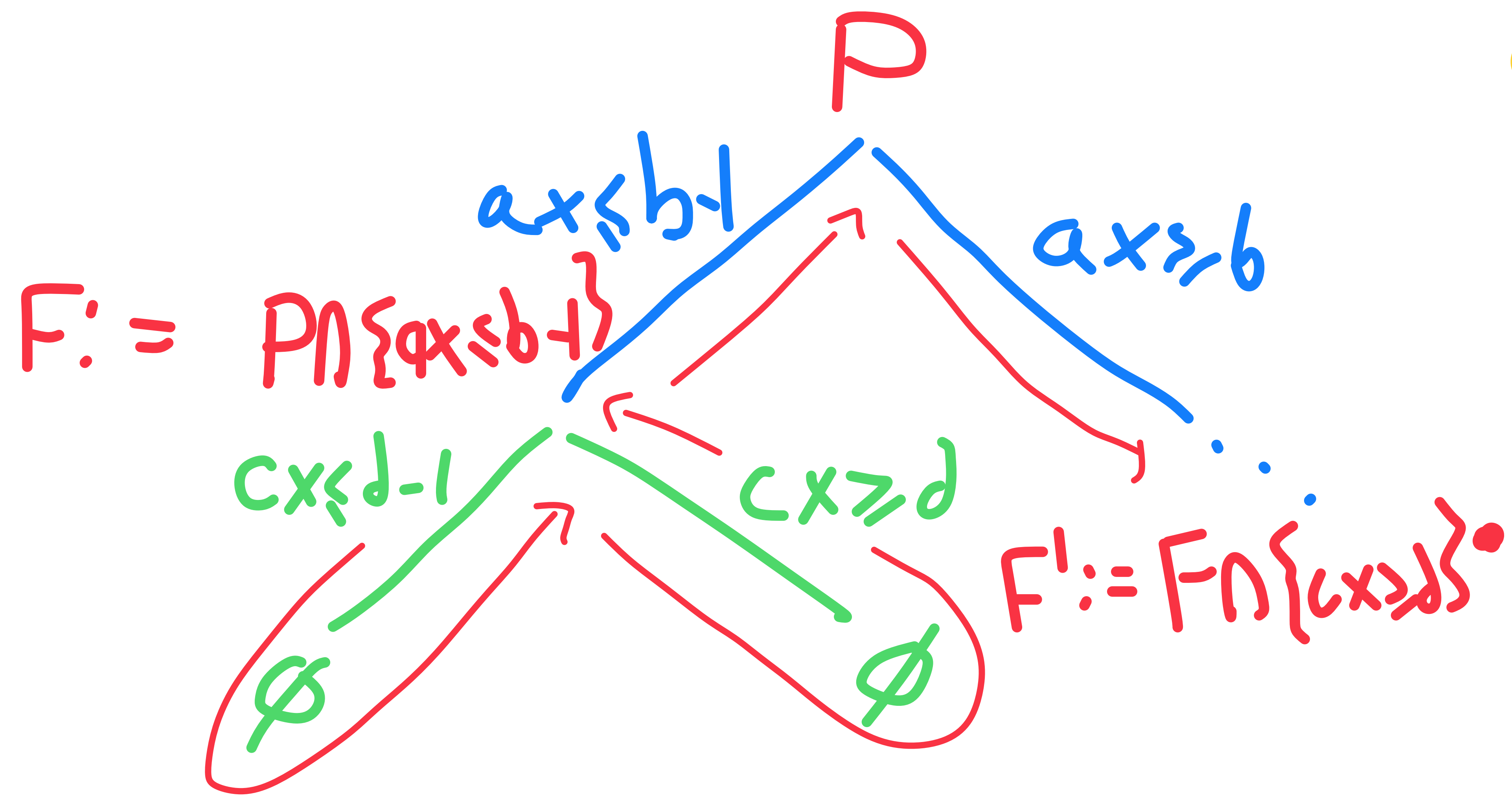

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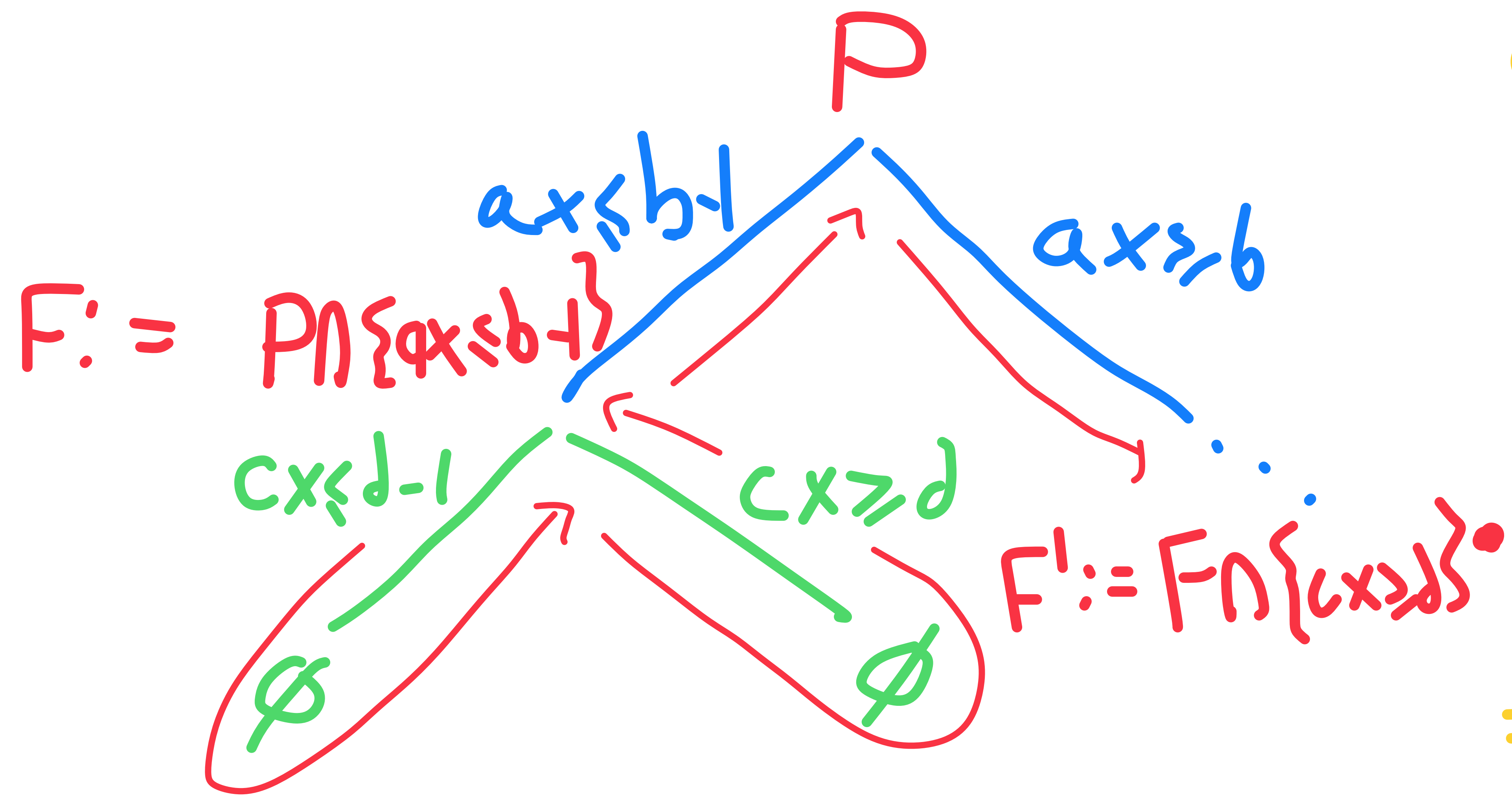

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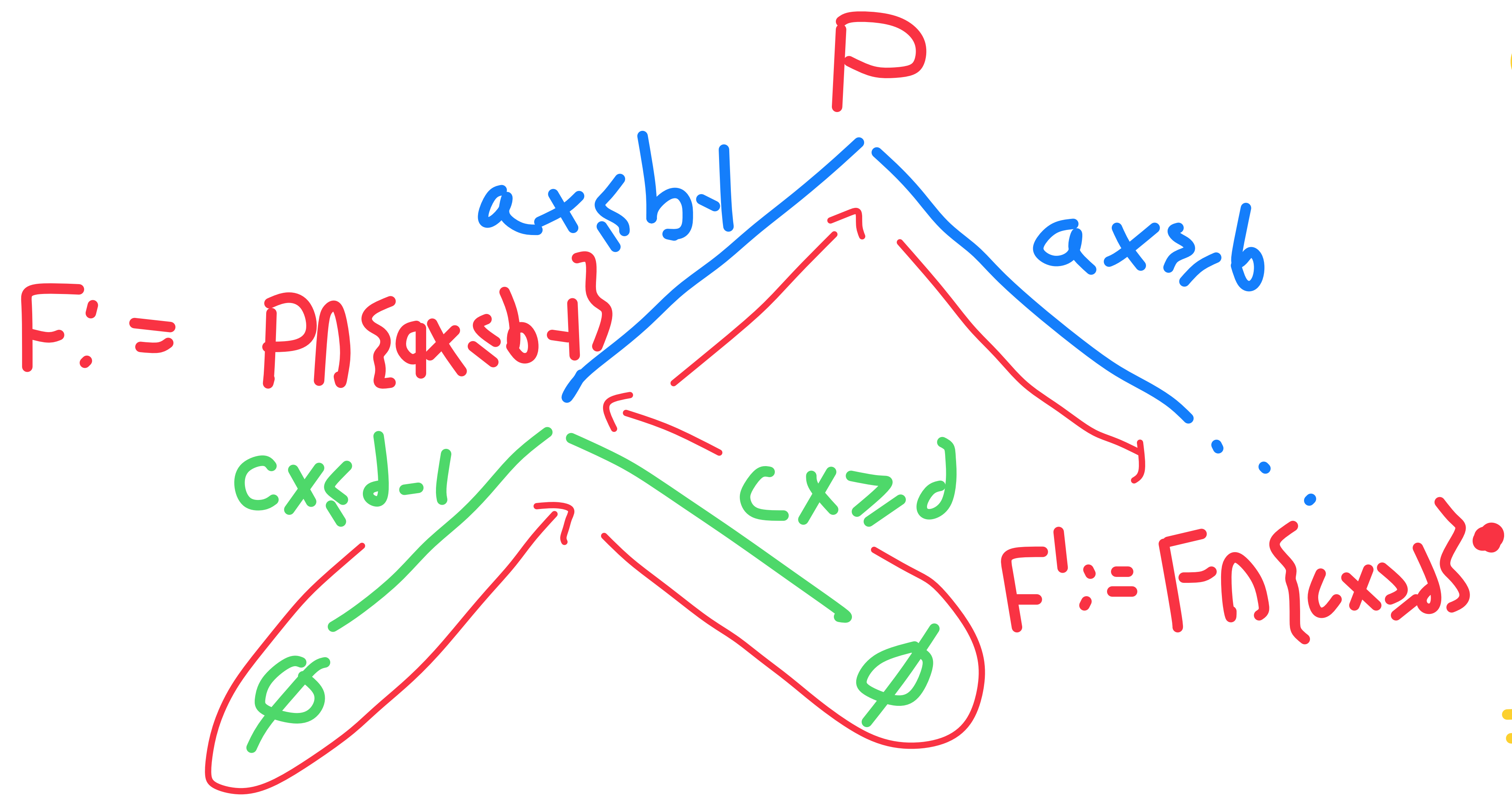

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$\Longrightarrow$ $(ax \leq b-1,\ ax \geq b)$ is pathlike for P'

$\Longrightarrow$ $P' \vdash P \cap \{ax \geq b\}$

Quasi-poly Simulation of SP*

Thm: Any SP proof of size s implies a size $s(cn)^{\log s}$ Facelike proof

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→ focus on polytopes in $[0,1]^n$, but holds for others.

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Proof Sketch: Fix an SP^* proof π.

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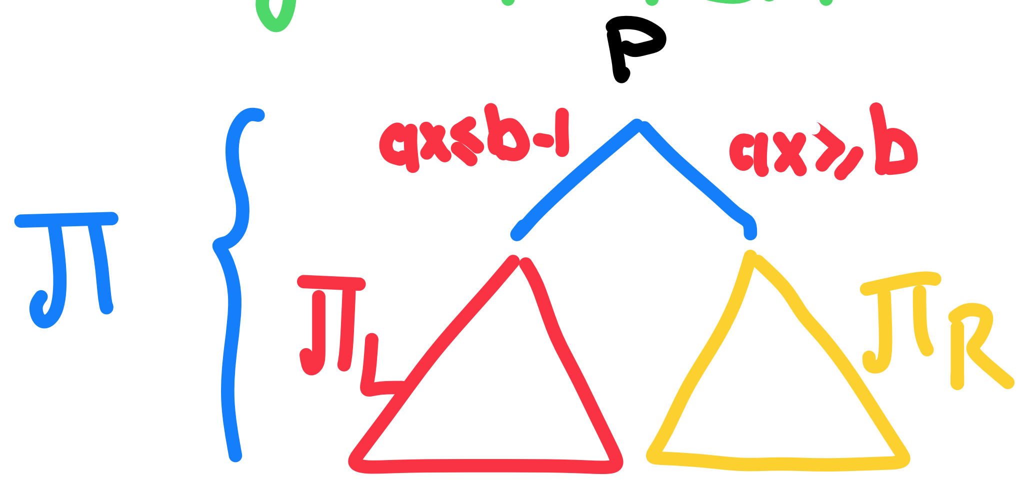

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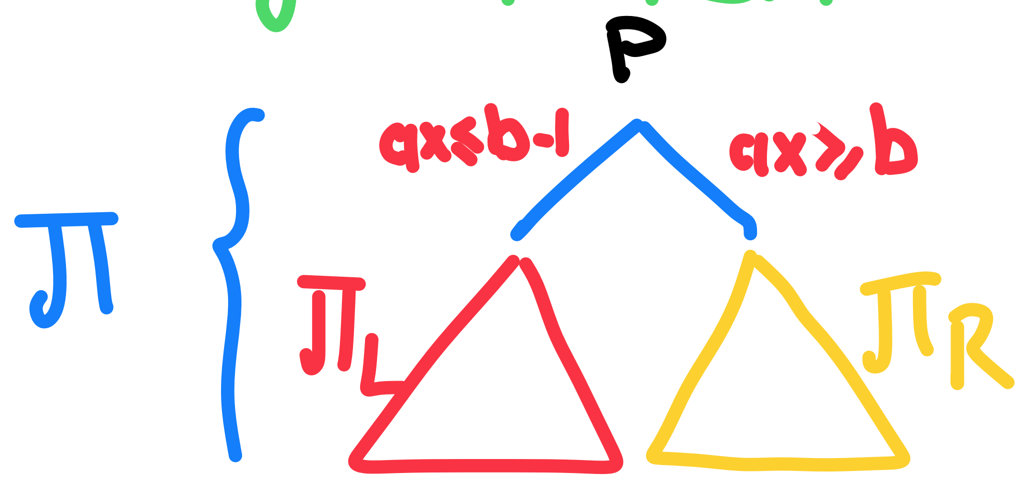

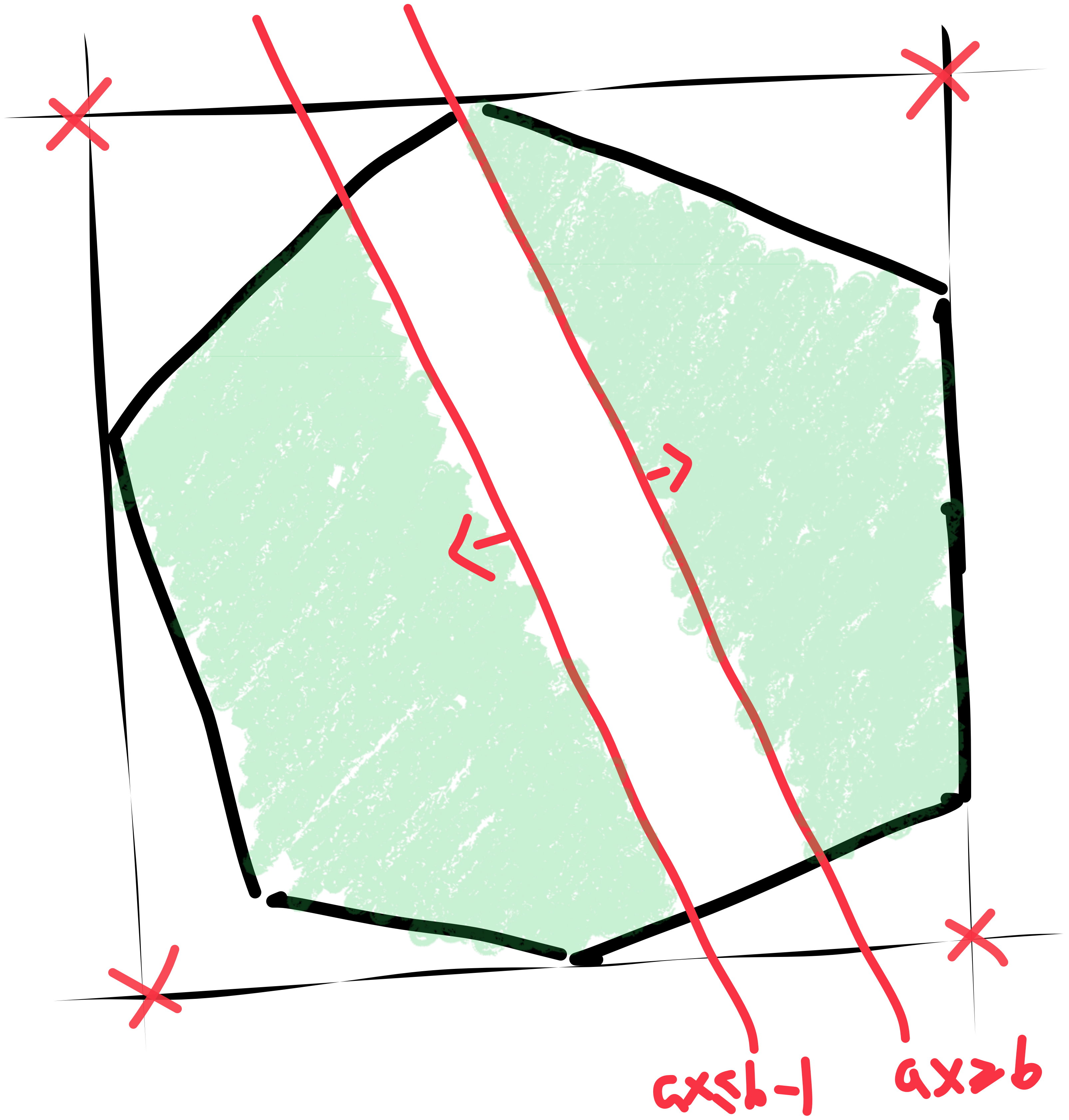

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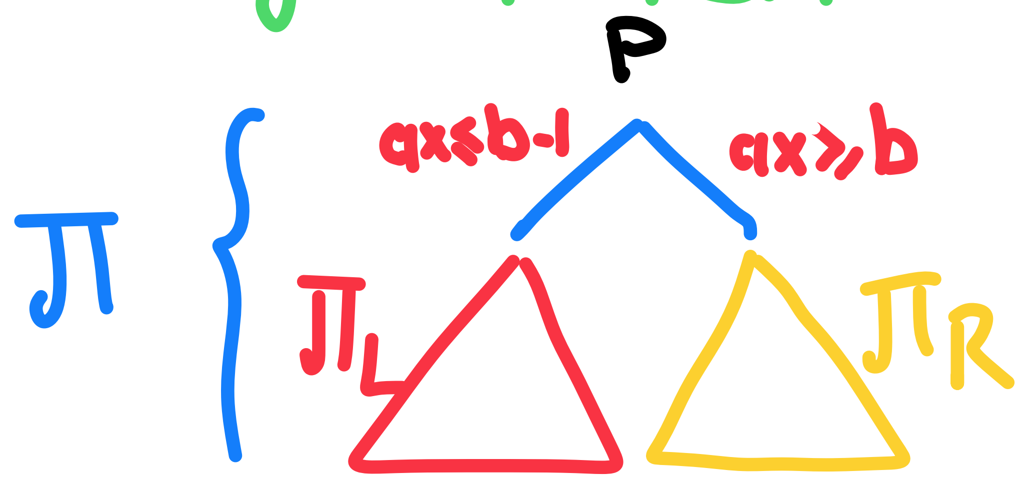
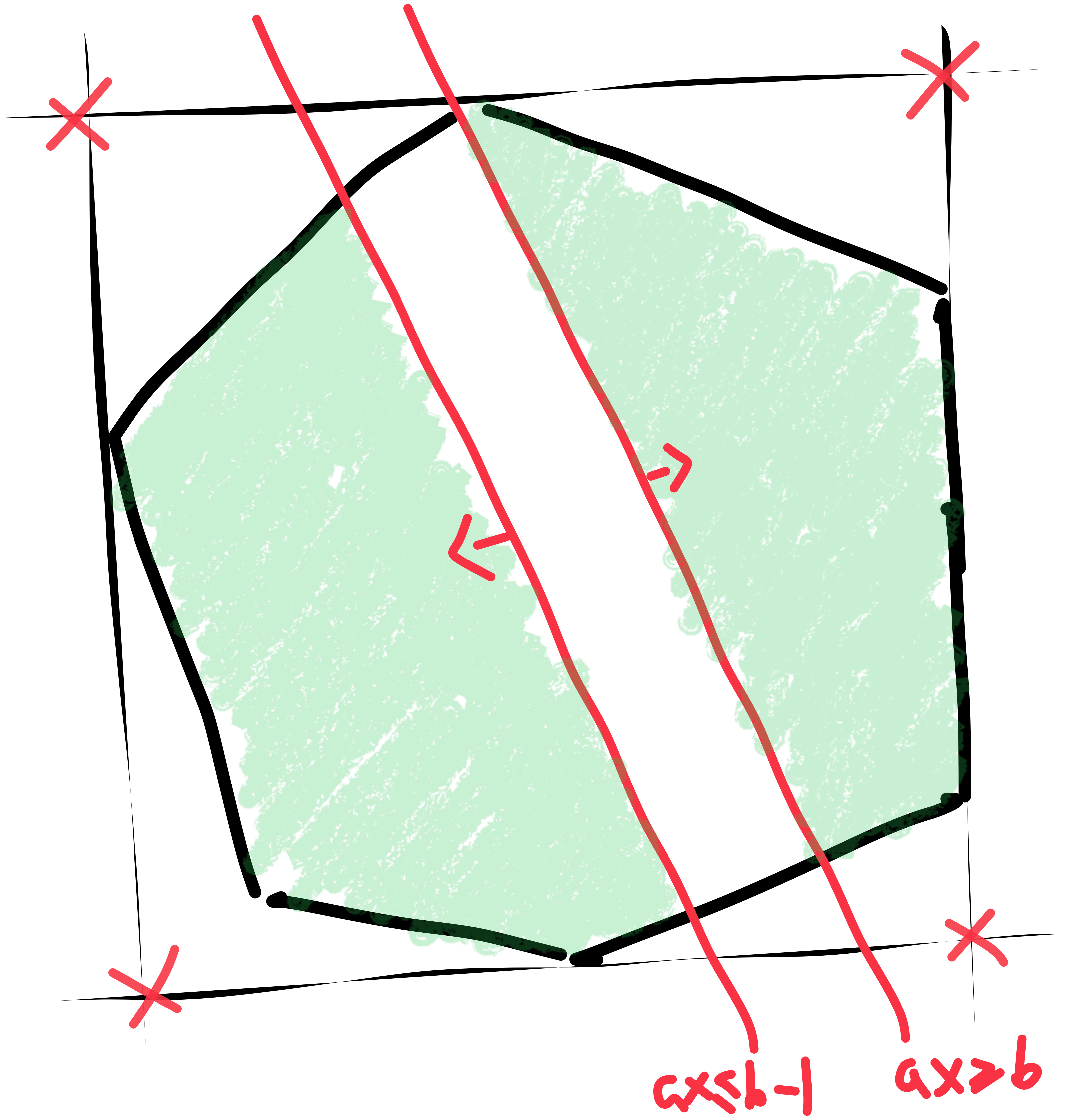


Suppose $|\pi_L| \leq |\pi_R|$



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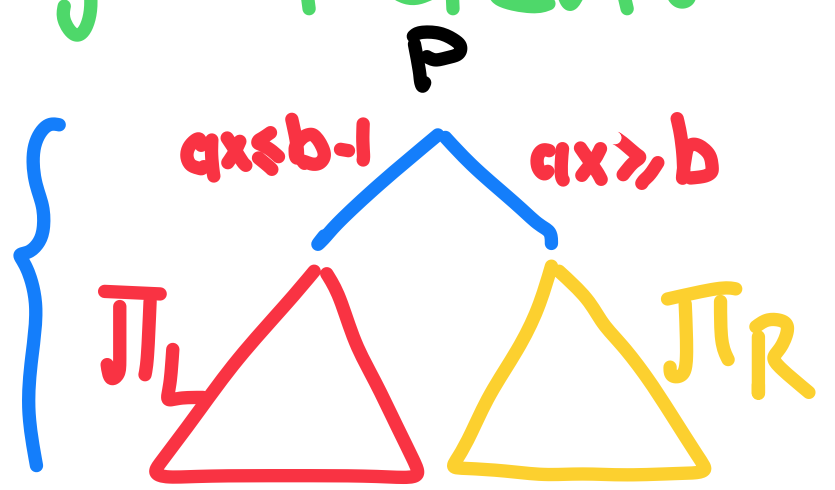
π 

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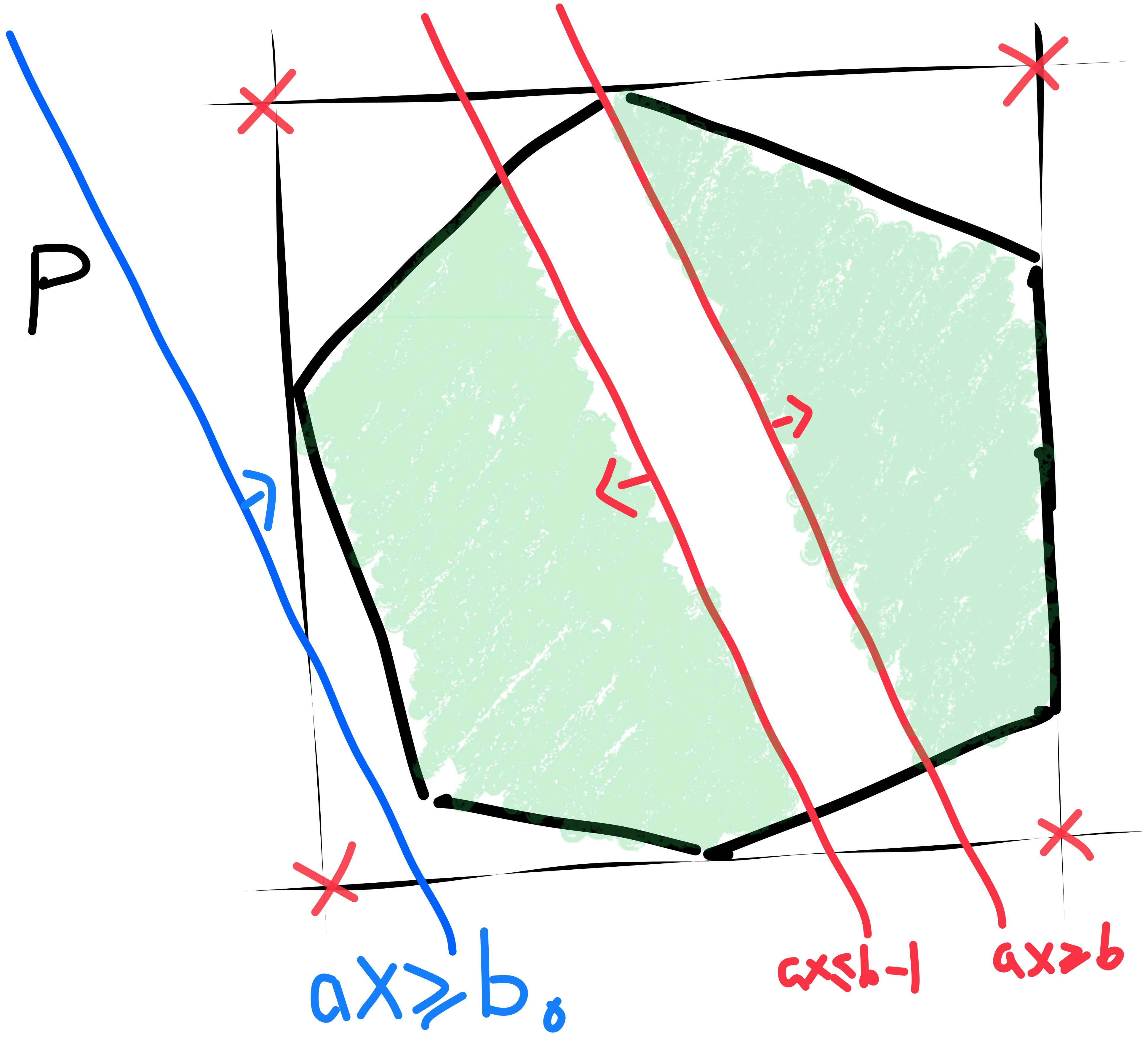
- Let b_0 be largest integer s.t. $ax \geq b_0$ valid for P



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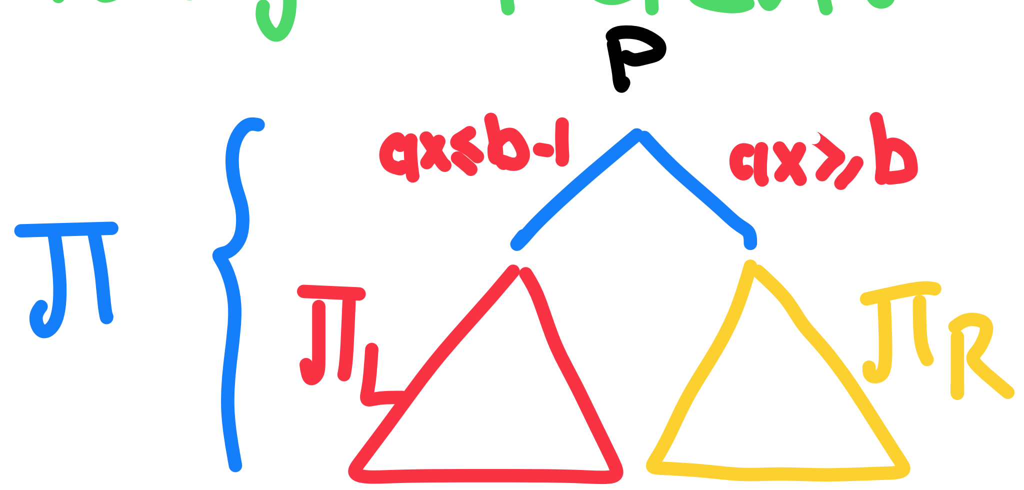
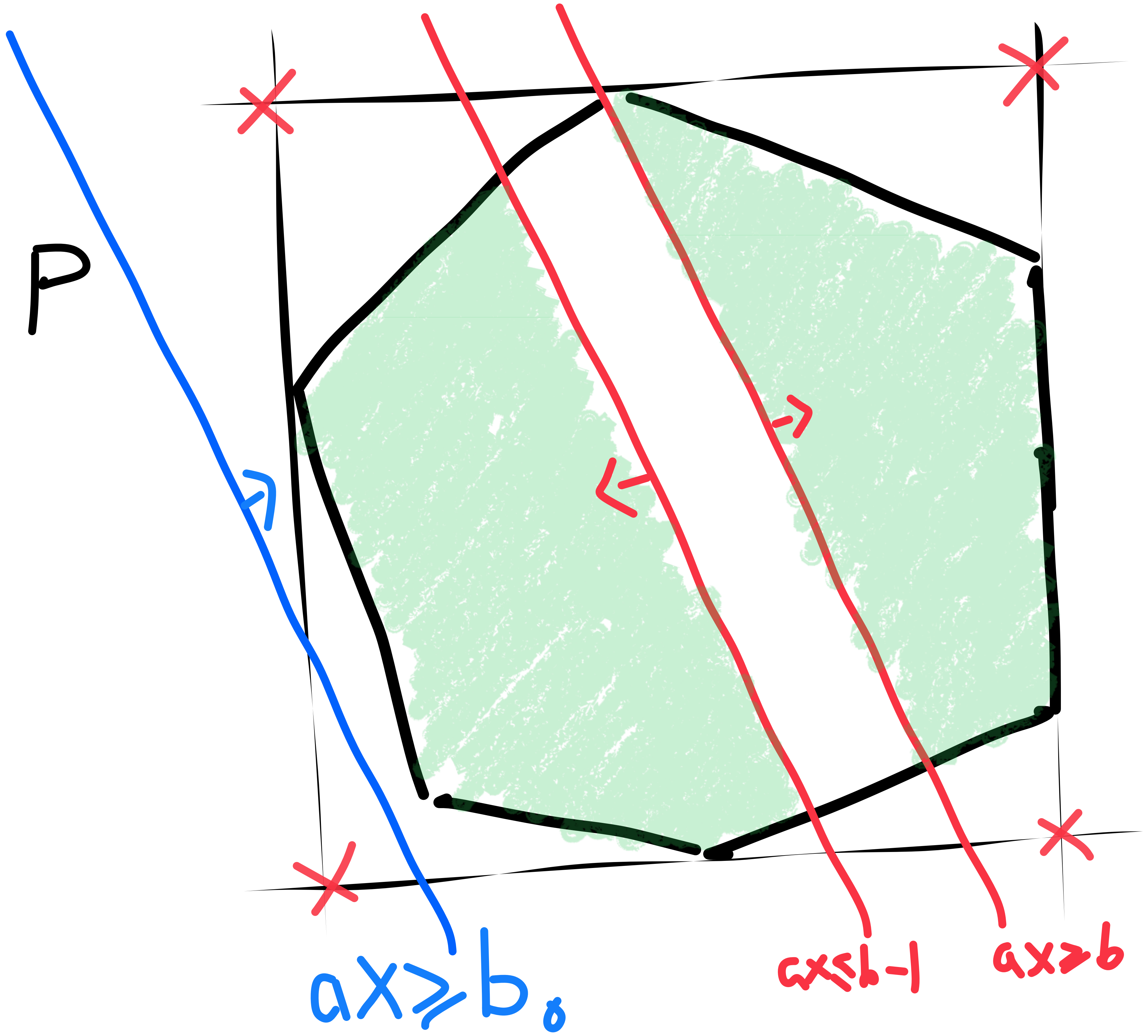


Suppose $|\pi_L| \le |\pi_R|$

- Let b_0 be largest integer s.t. $ax \ge b_0$ valid for P
- Query $(ax \le b_0 - 1 + i,\ ax \ge b_0 + i)$ for $i = 0, \dots, b - b_0$



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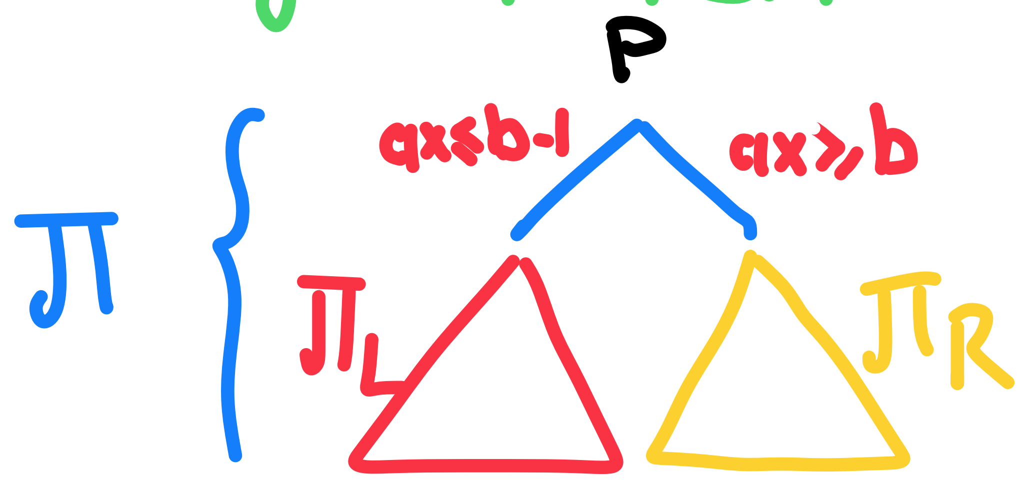

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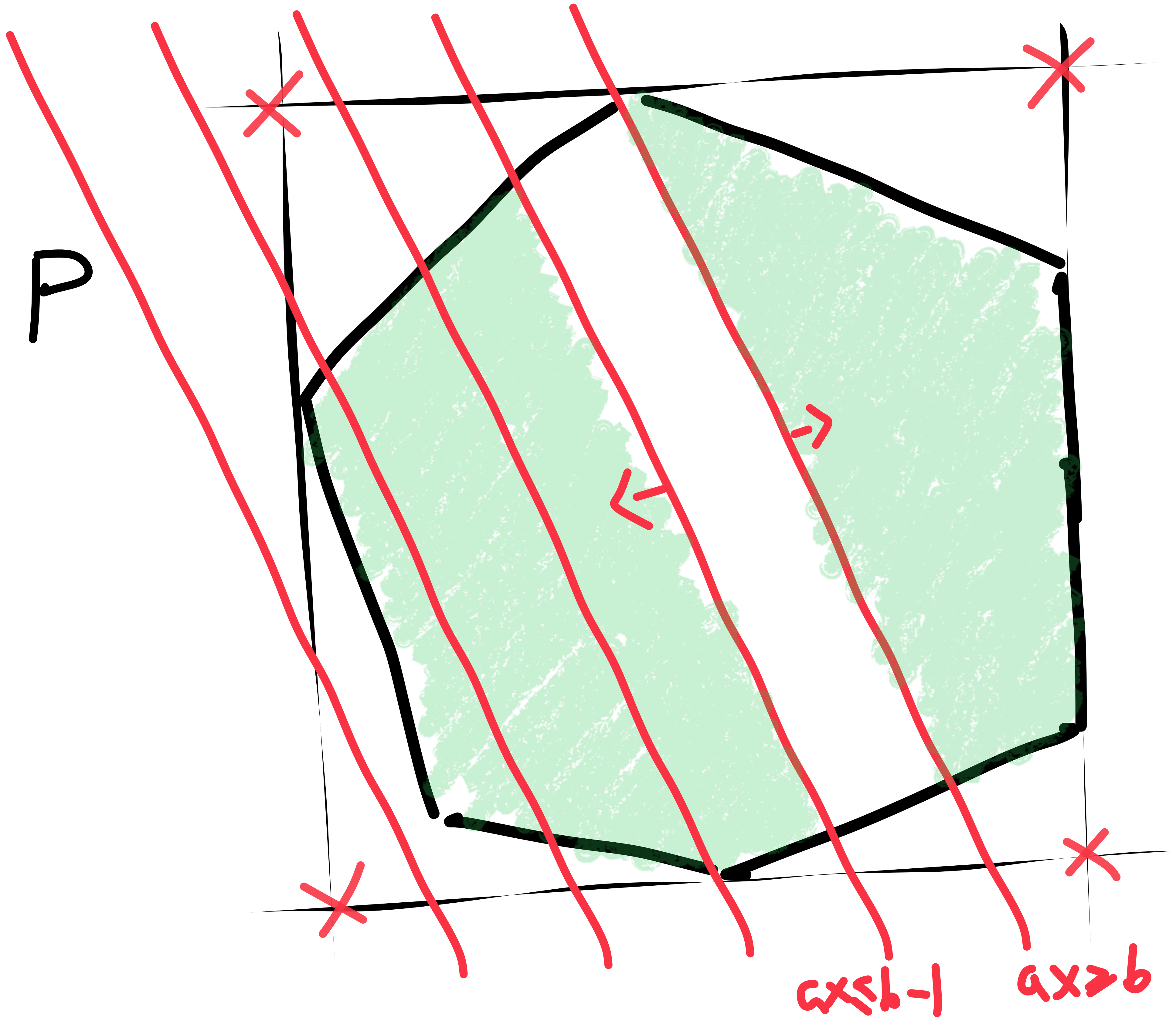

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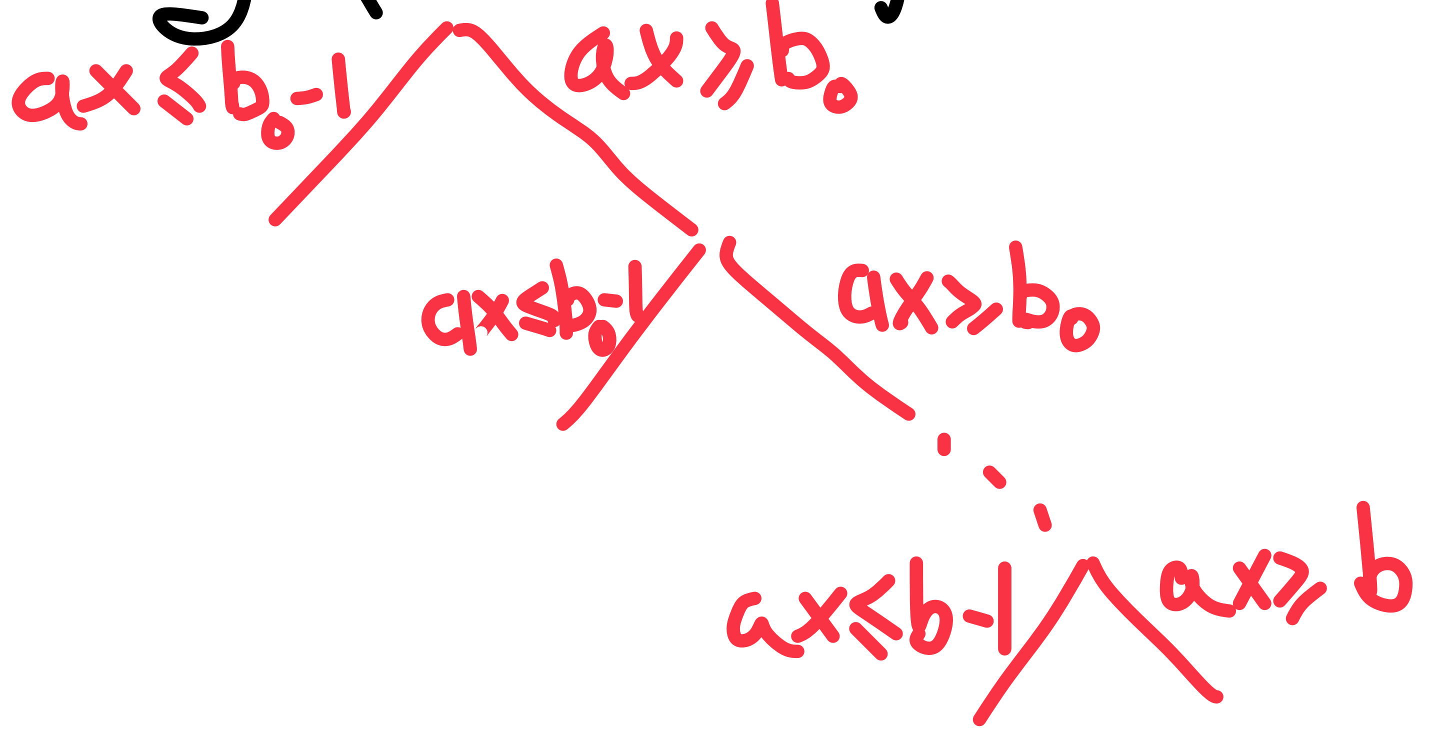

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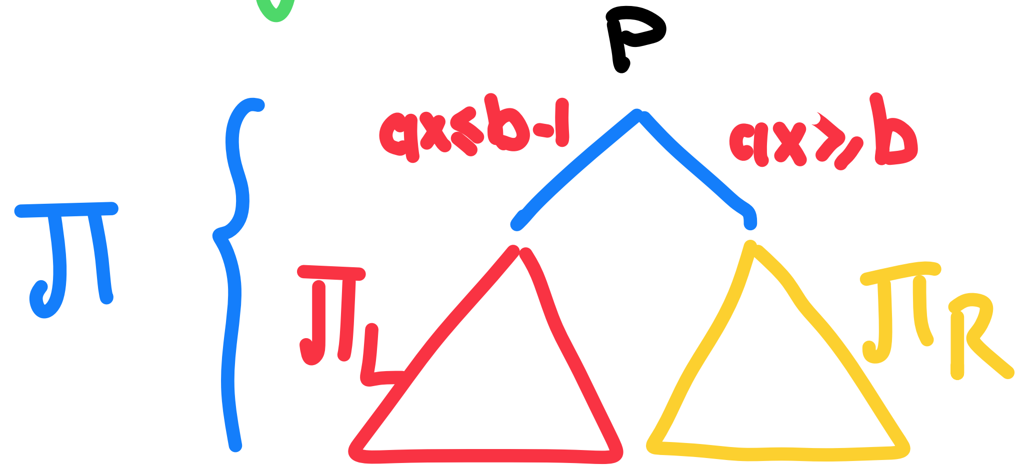

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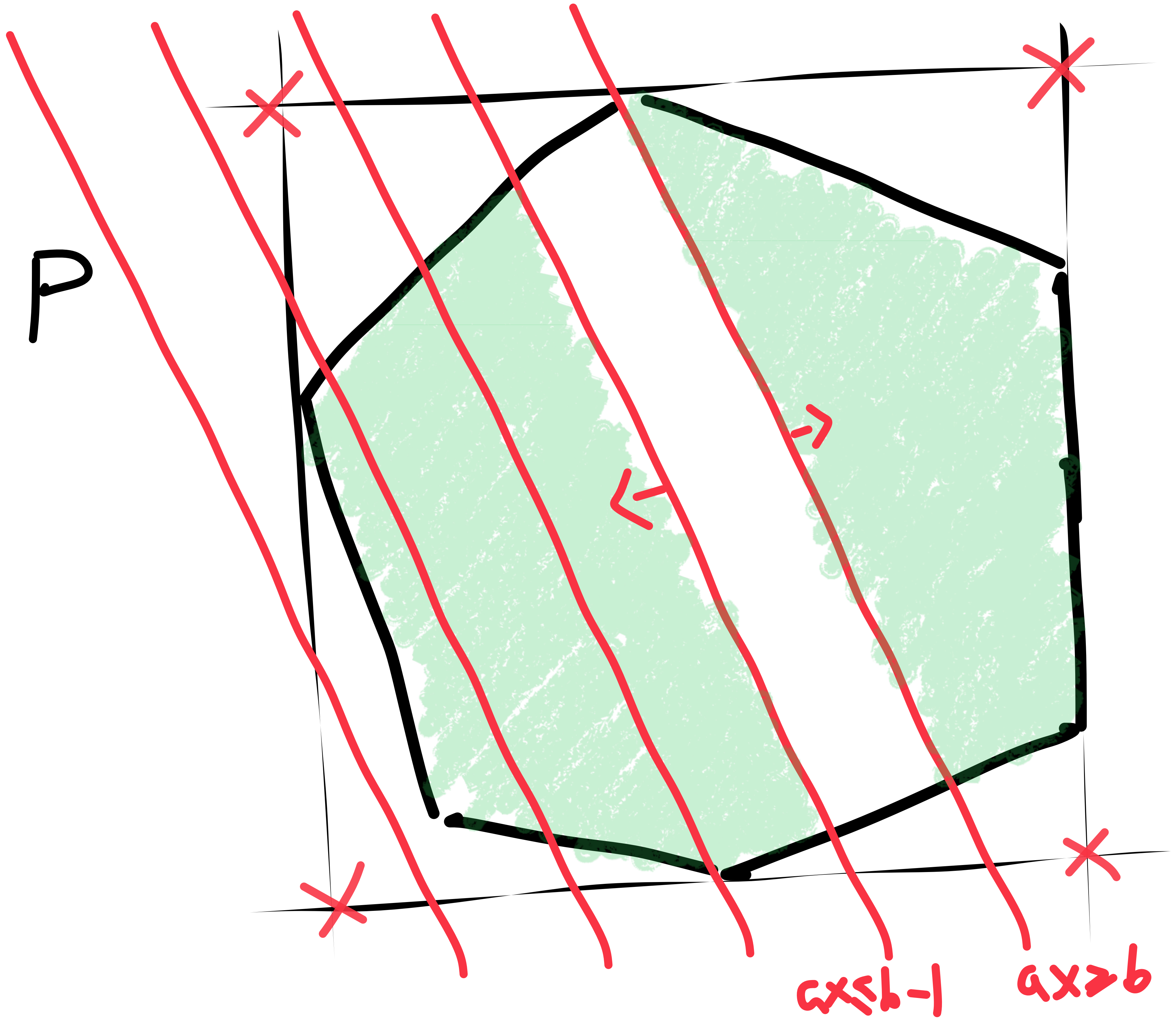

- Let b_0 be largest integer s.t. $ax \ge b_0$ valid for P
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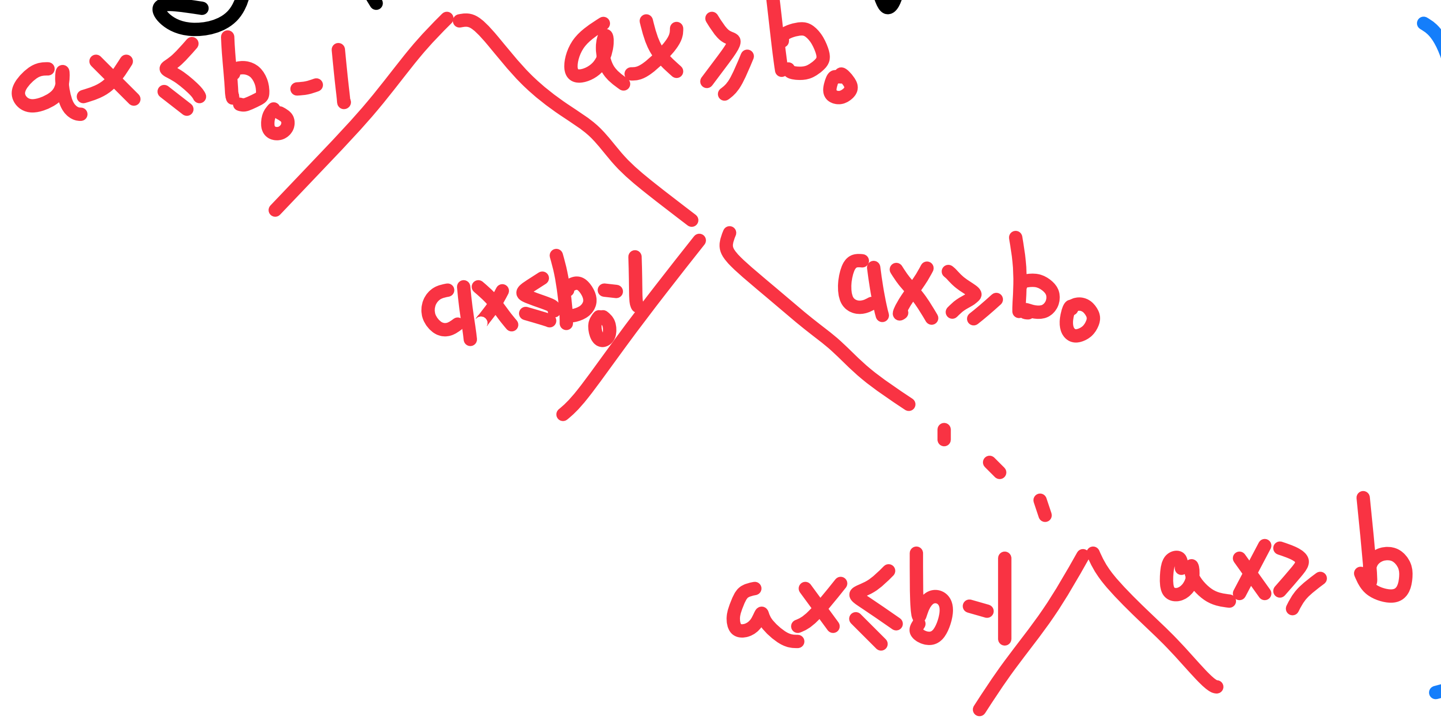

Facelike

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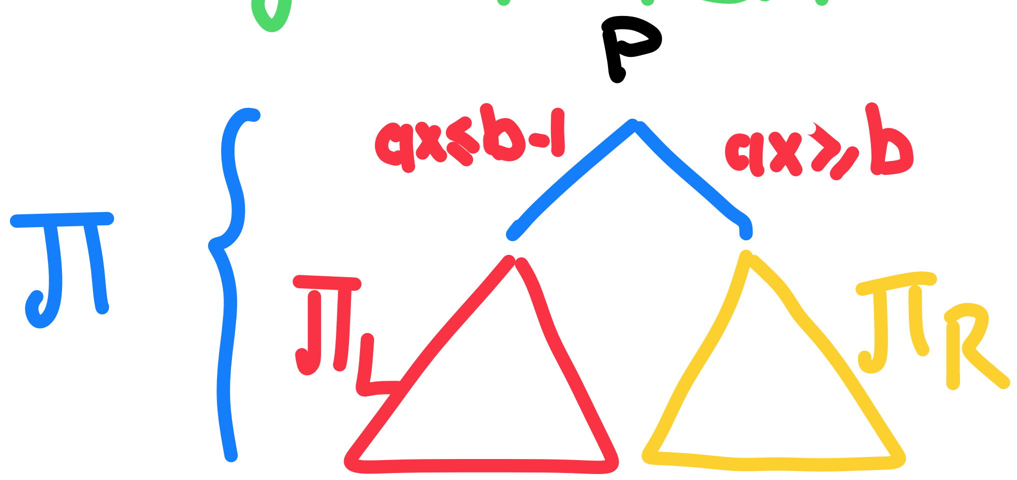

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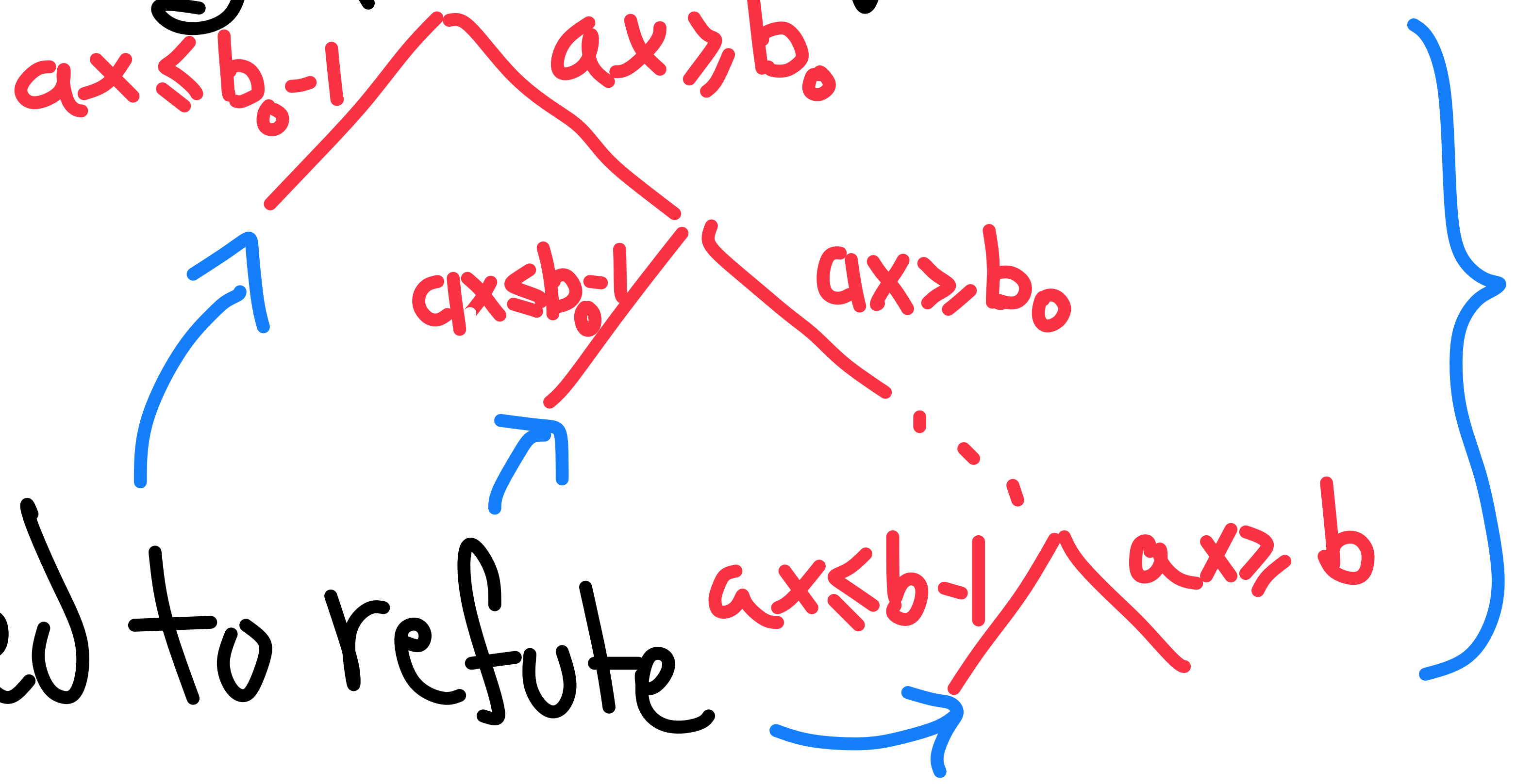
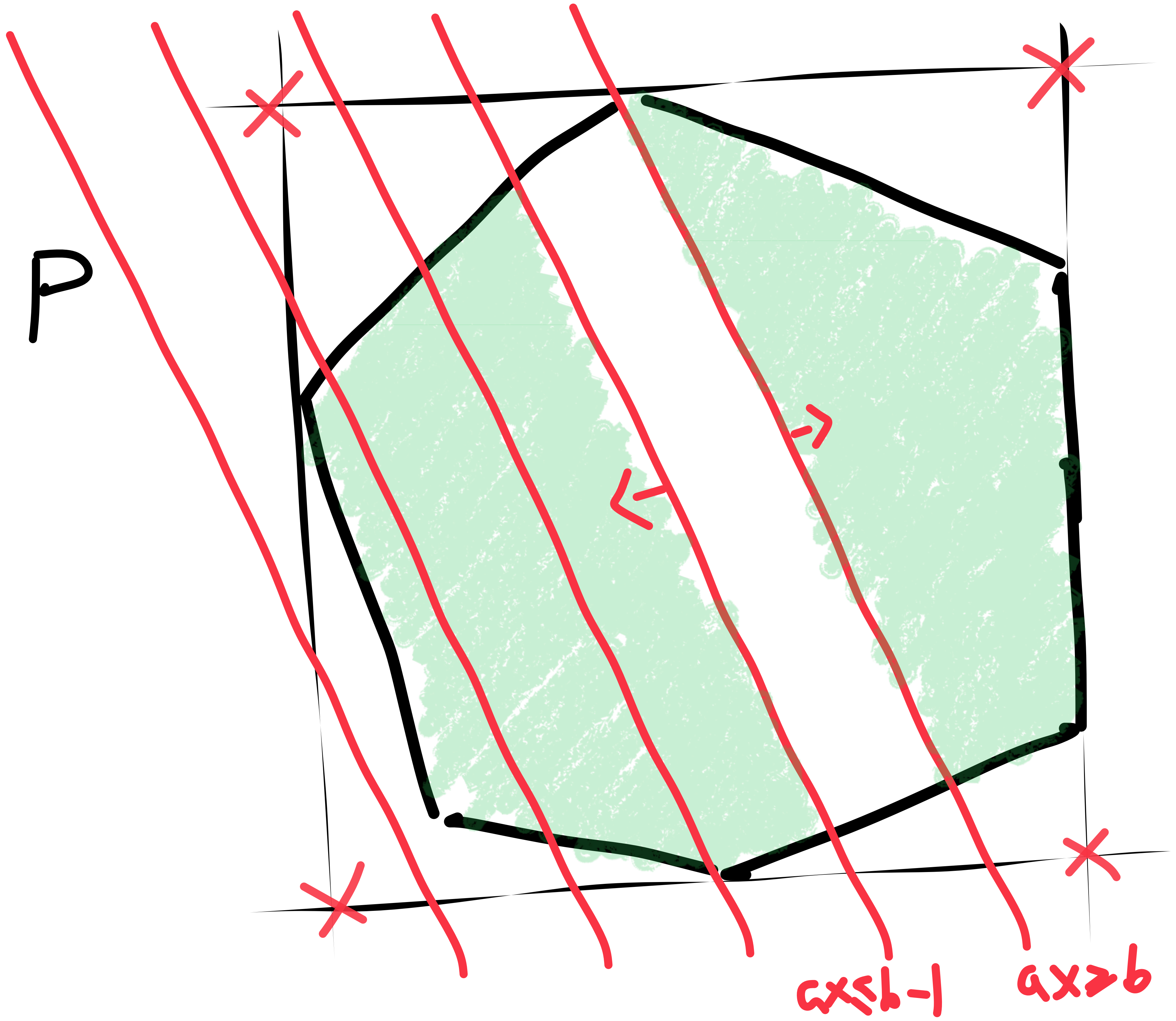


Need to refute



Quasi-poly Simulation of SP*

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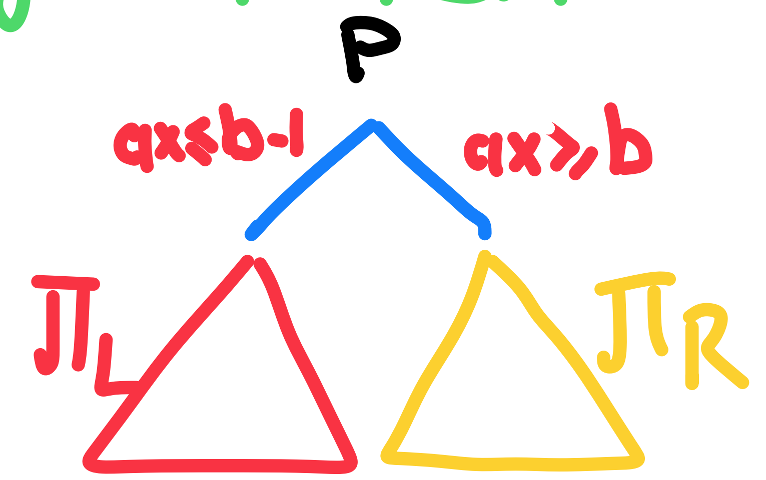
Proof Sketch: Fix an SP* proof π.

π 

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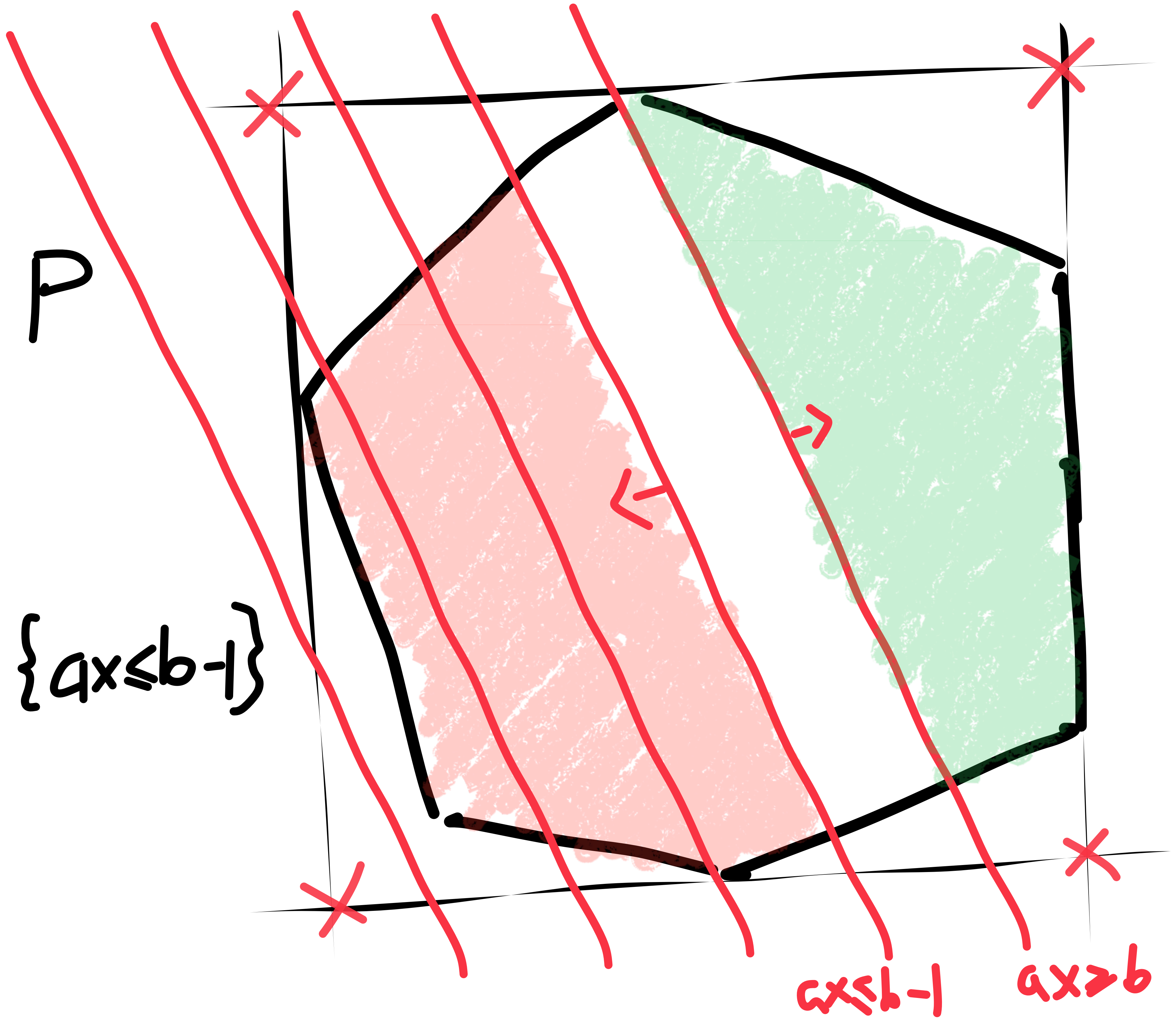


- Let b_0 be largest integer s.t. $ax \geq b_0$ valid for P
- Query $(ax \leq b_0 - 1 + i,\ ax \geq b_0 + i)$ for $i = 0, \ldots, b - b_0$

$ax \leq b_0 - 1$ $\quad$ $ax \geq b_0$

$ax \leq b_0 - 1$ $\quad$ $ax \geq b_0$

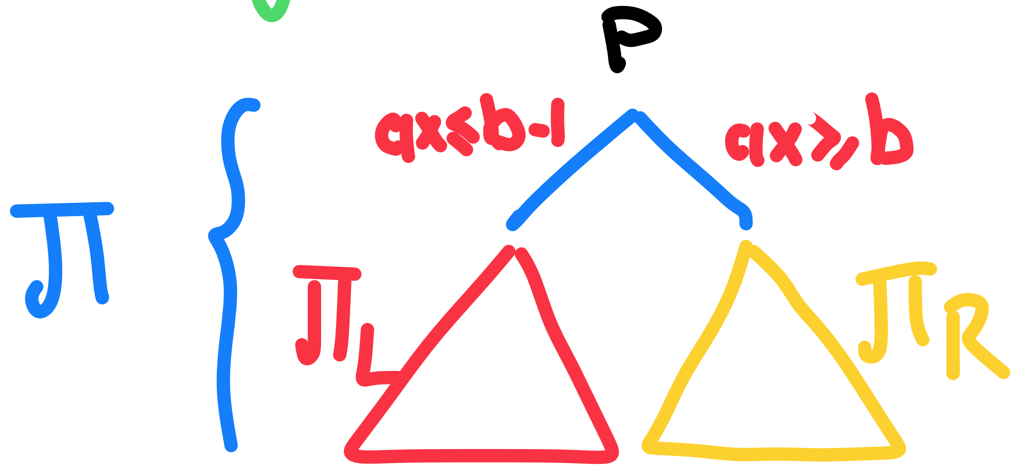
Need to refute $\quad ax \leq b - 1 \quad ax \geq b$

Observe: π_L refutes $P \cap \{ax \leq b-1\}$

Quasi-poly Simulation of SP^*

Thm: Any SP proof of size s implies a size $s(cn)^{\log s}$ Facelike proof

Proof Sketch: Fix an SP^* proof π.



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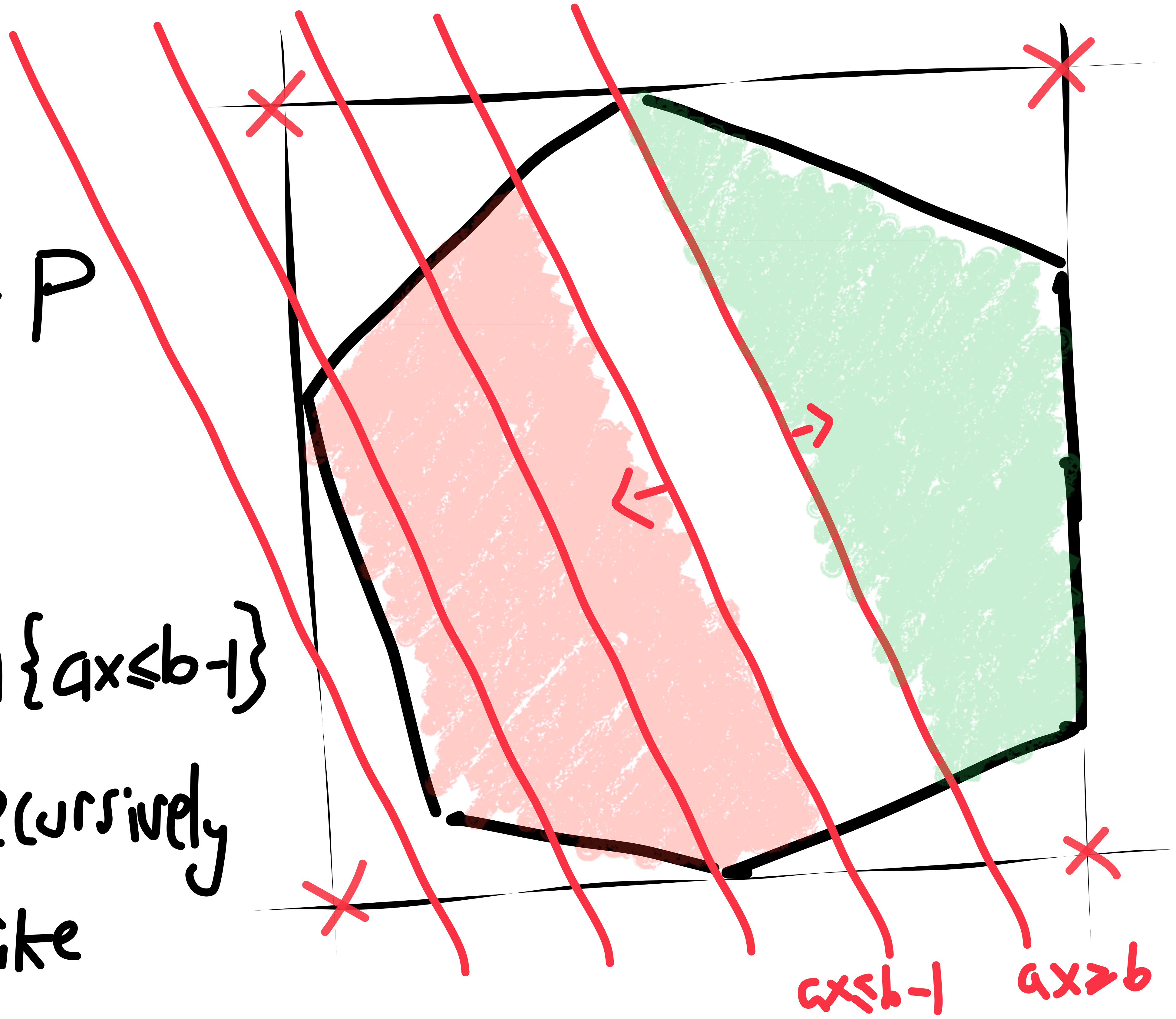

- Let b_0 be largest integer s.t. $ax \ge b_0$ valid for P
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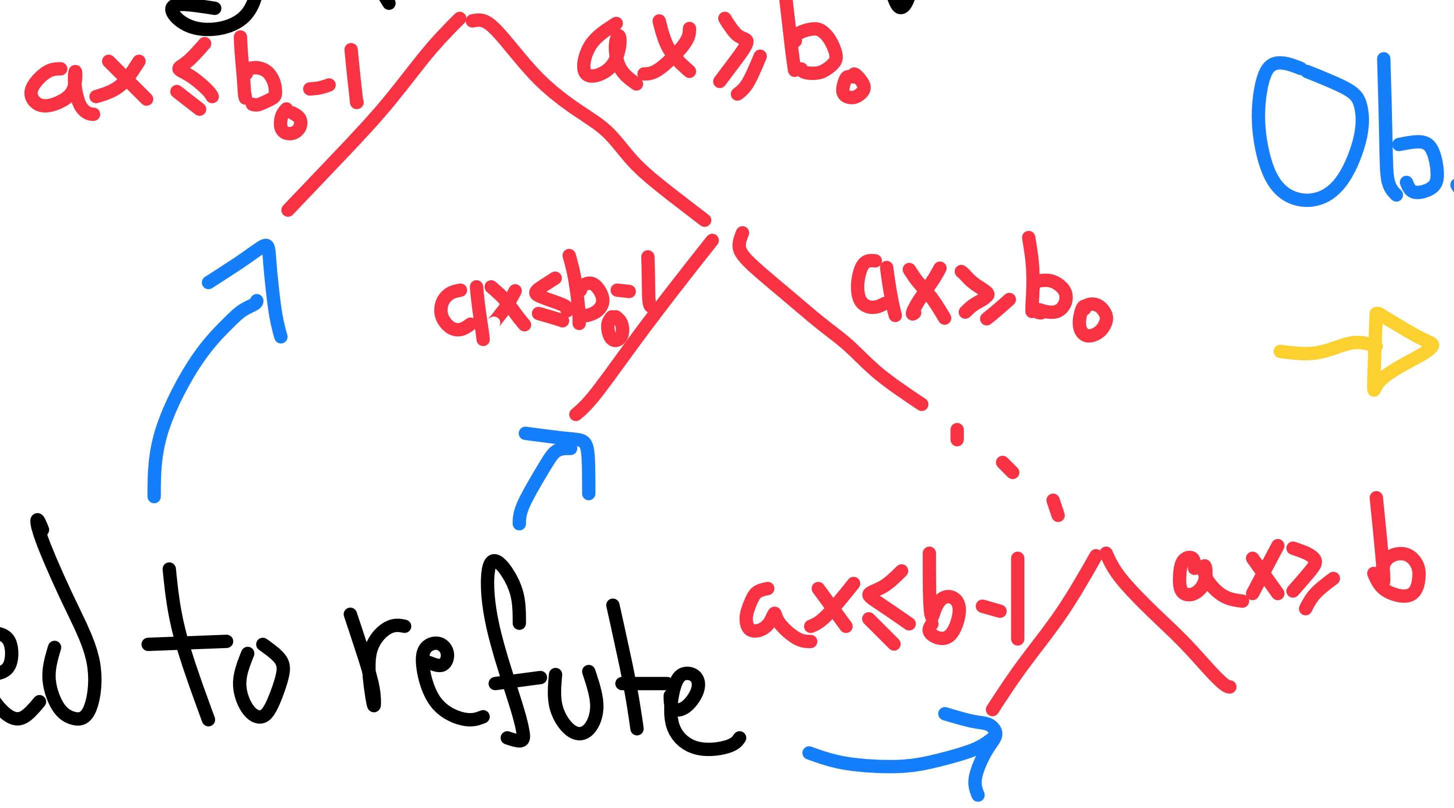

Need to refute

Observe: π_L refutes $P \cap \{ax \le b-1\}$

→ Let π_L' be π_L recursively converted a facelike proof

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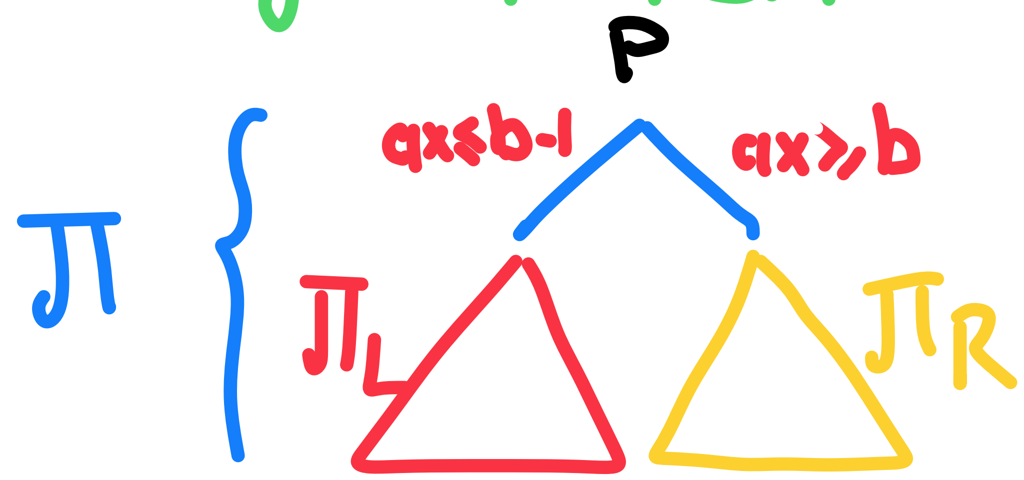
Proof Sketch: Fix an SP^* proof π.



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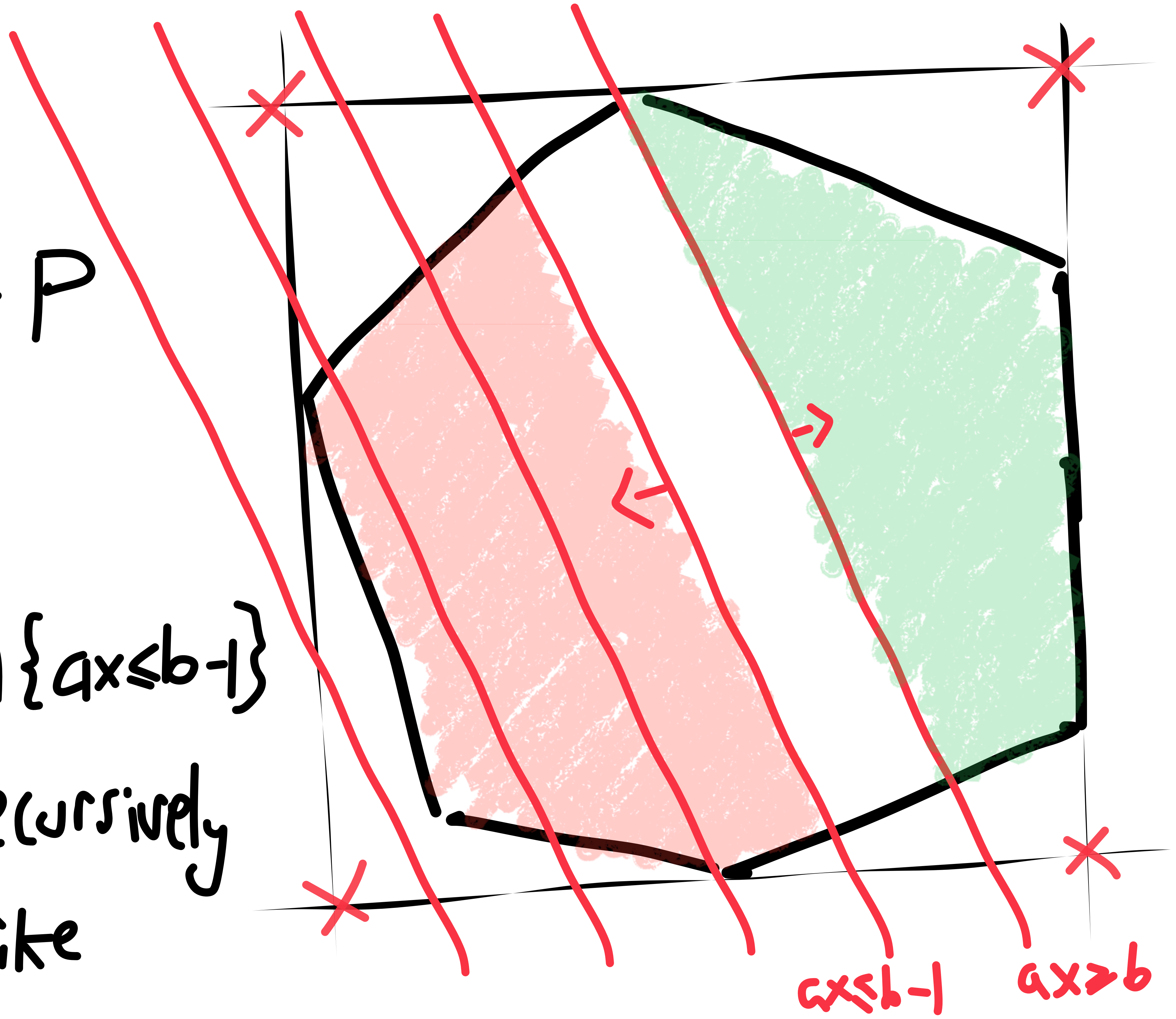
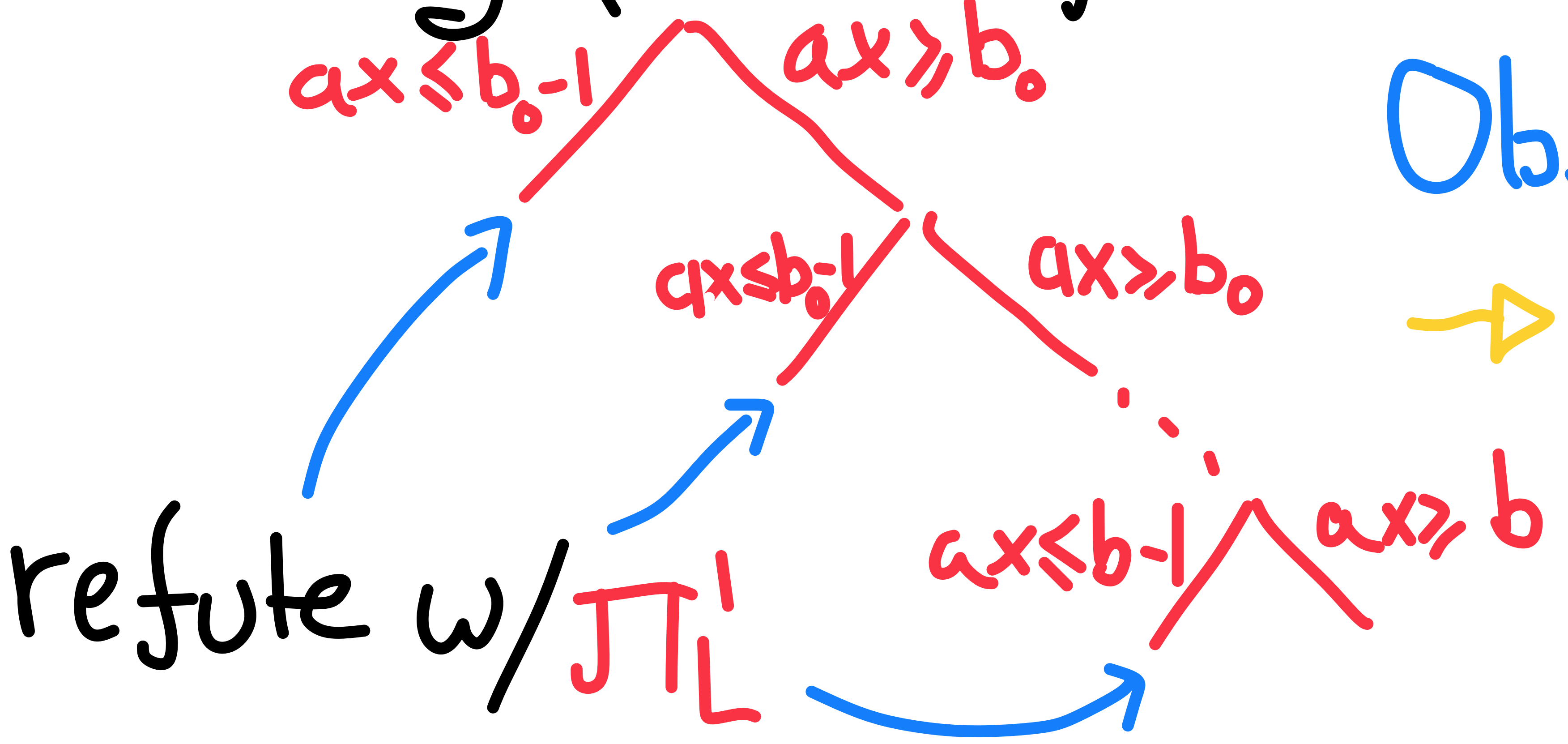


- Let b_0 be largest integer s.t. $ax \geq b_0$ valid for P
- Query $(ax \leq b_0 - 1 + i,\ ax \geq b_0 + i)$ for $i = 0, \ldots, b - b_0$



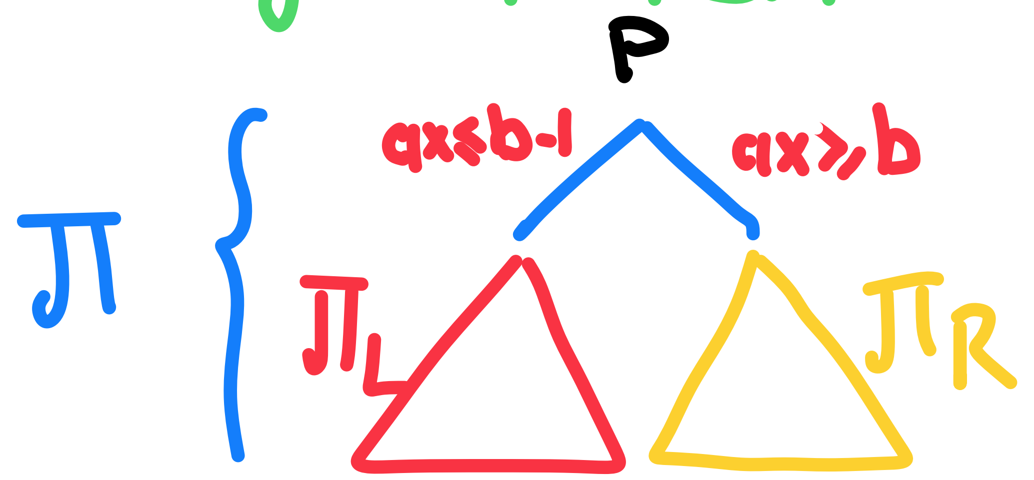
Observe: π_L refutes $P \cap \{ax \leq b-1\}$

→ Let π_L' be π_L recursively converted a facelike proof

Quasi-poly Simulation of SP*

Thm: Any SP proof of size s implies a size $s(cn)^{\log s}$ Facelike proof

Proof Sketch: Fix an SP* proof π.



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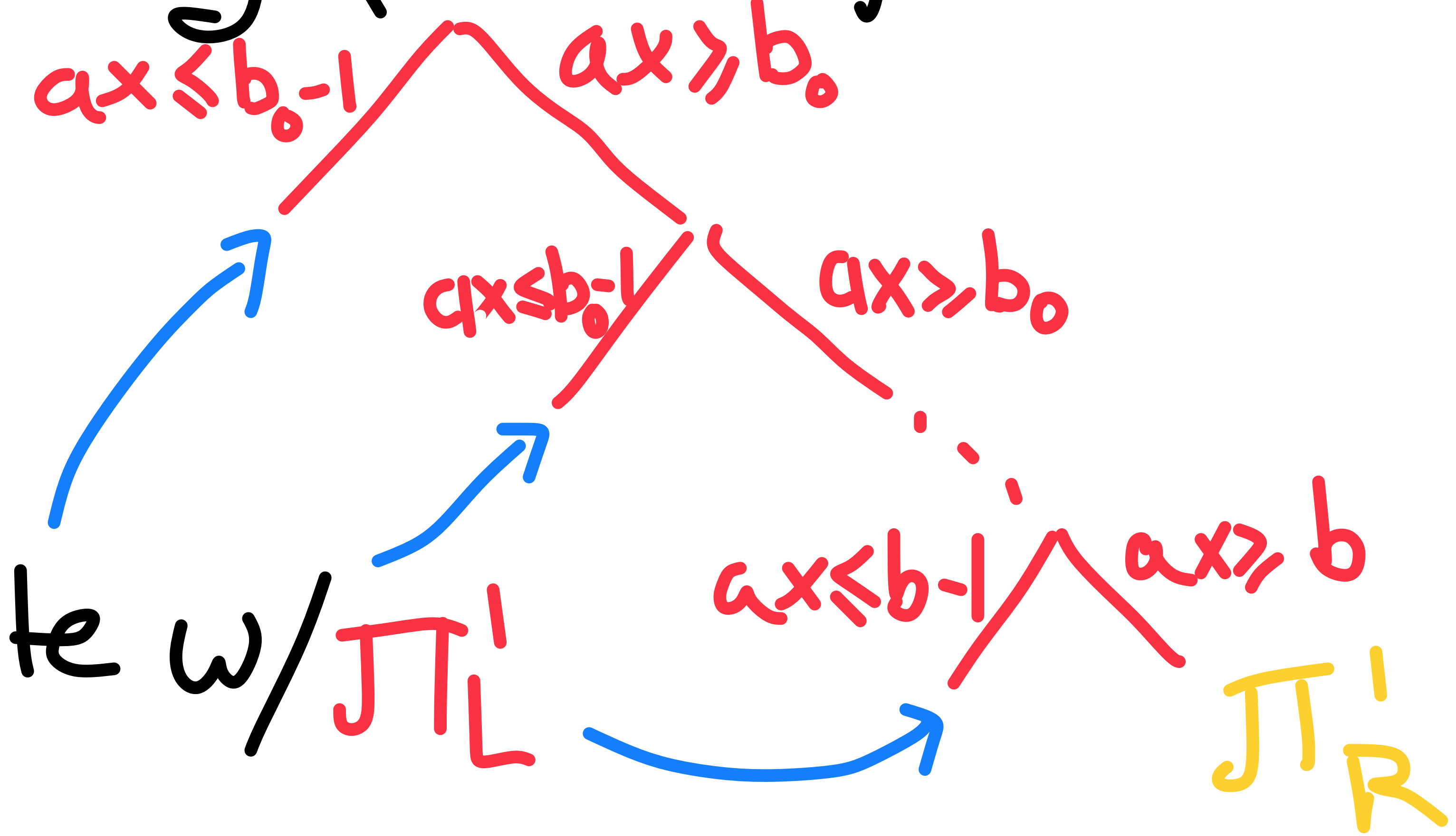

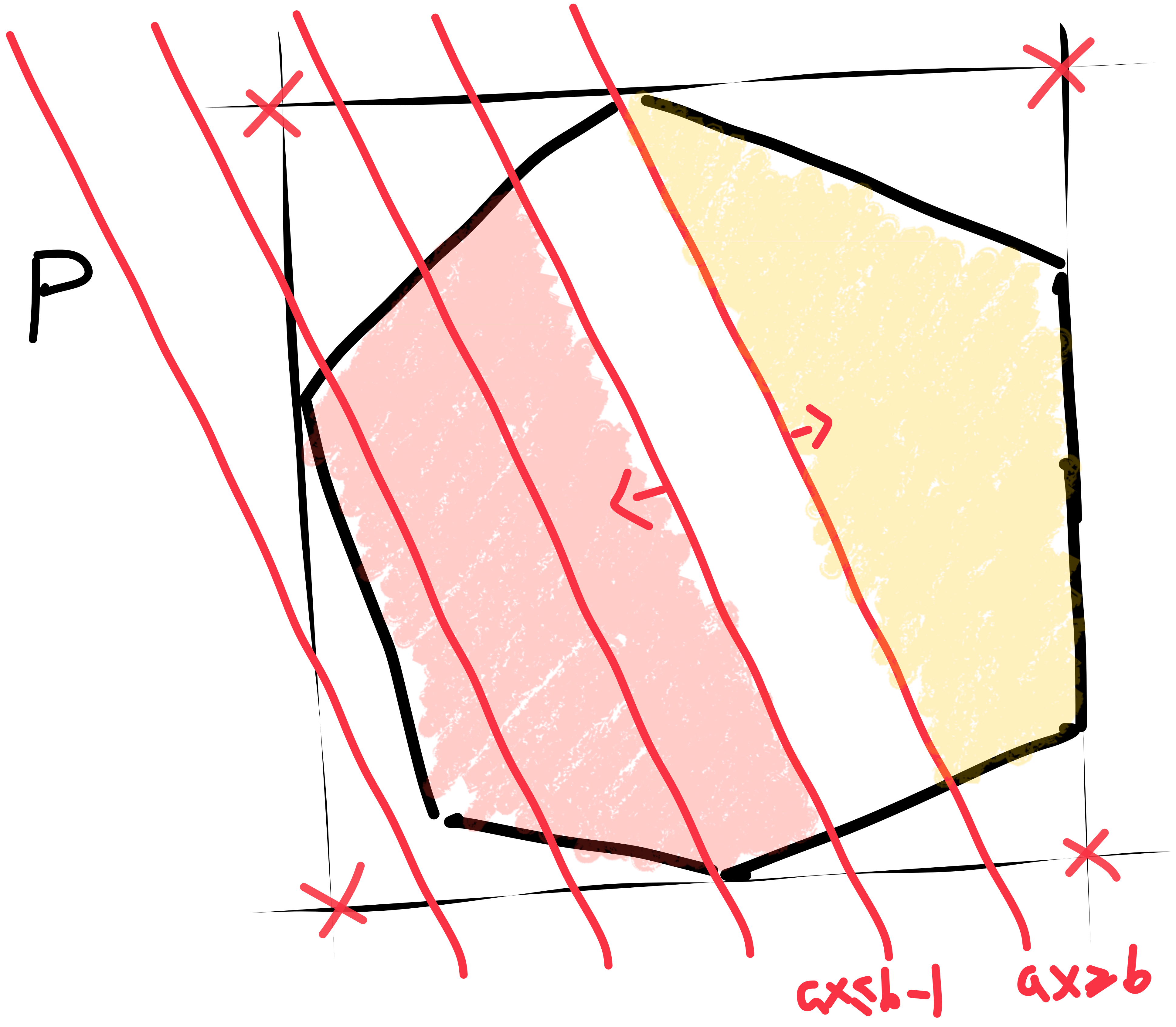

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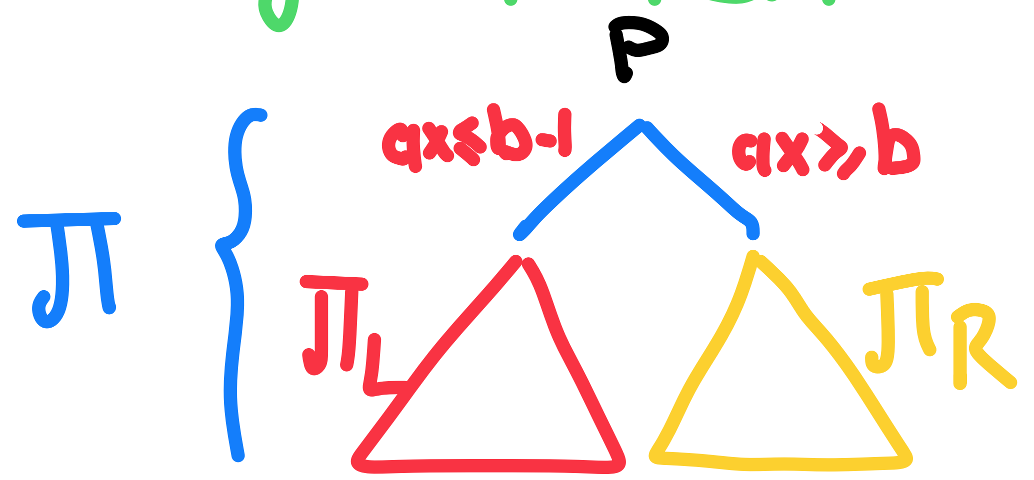

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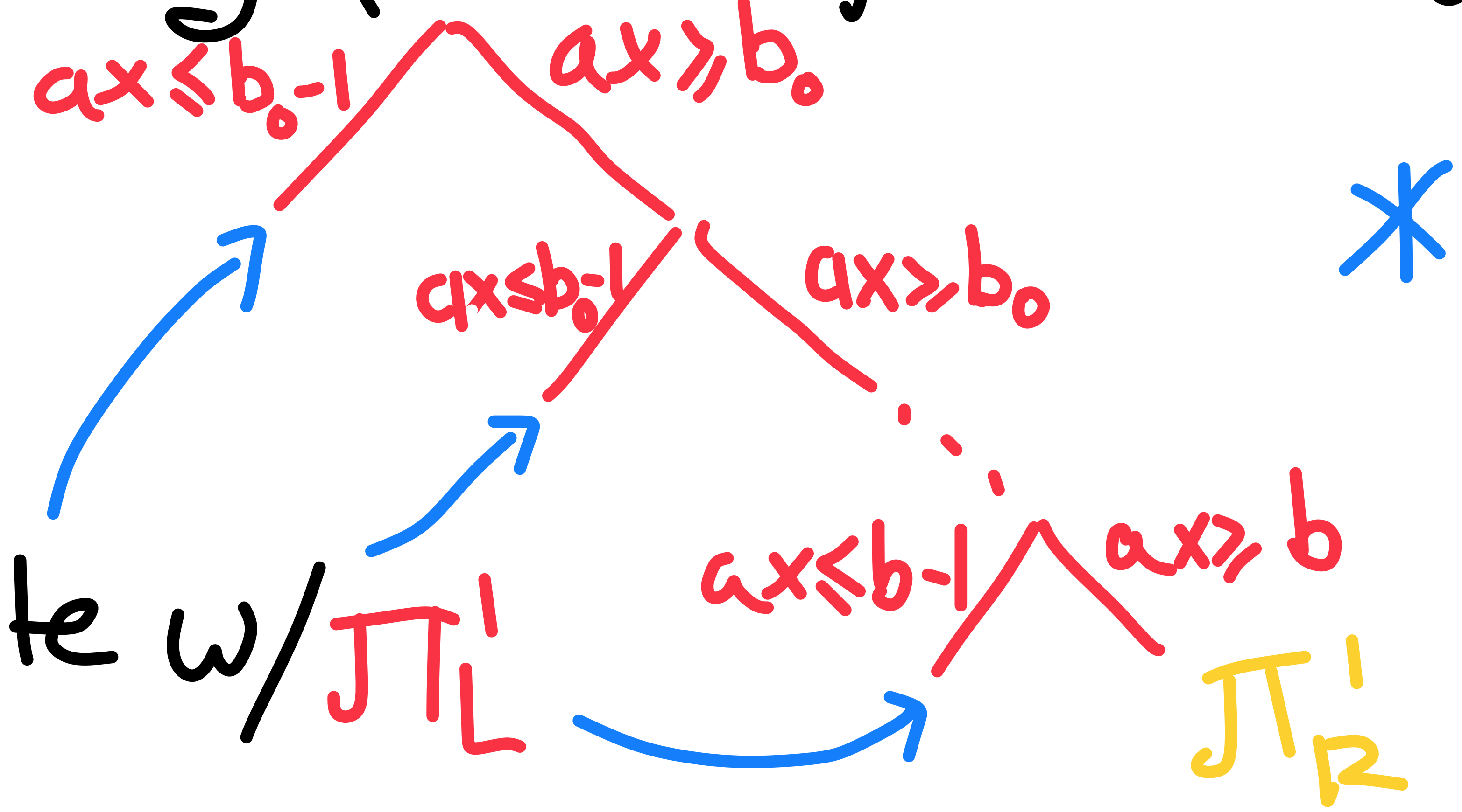
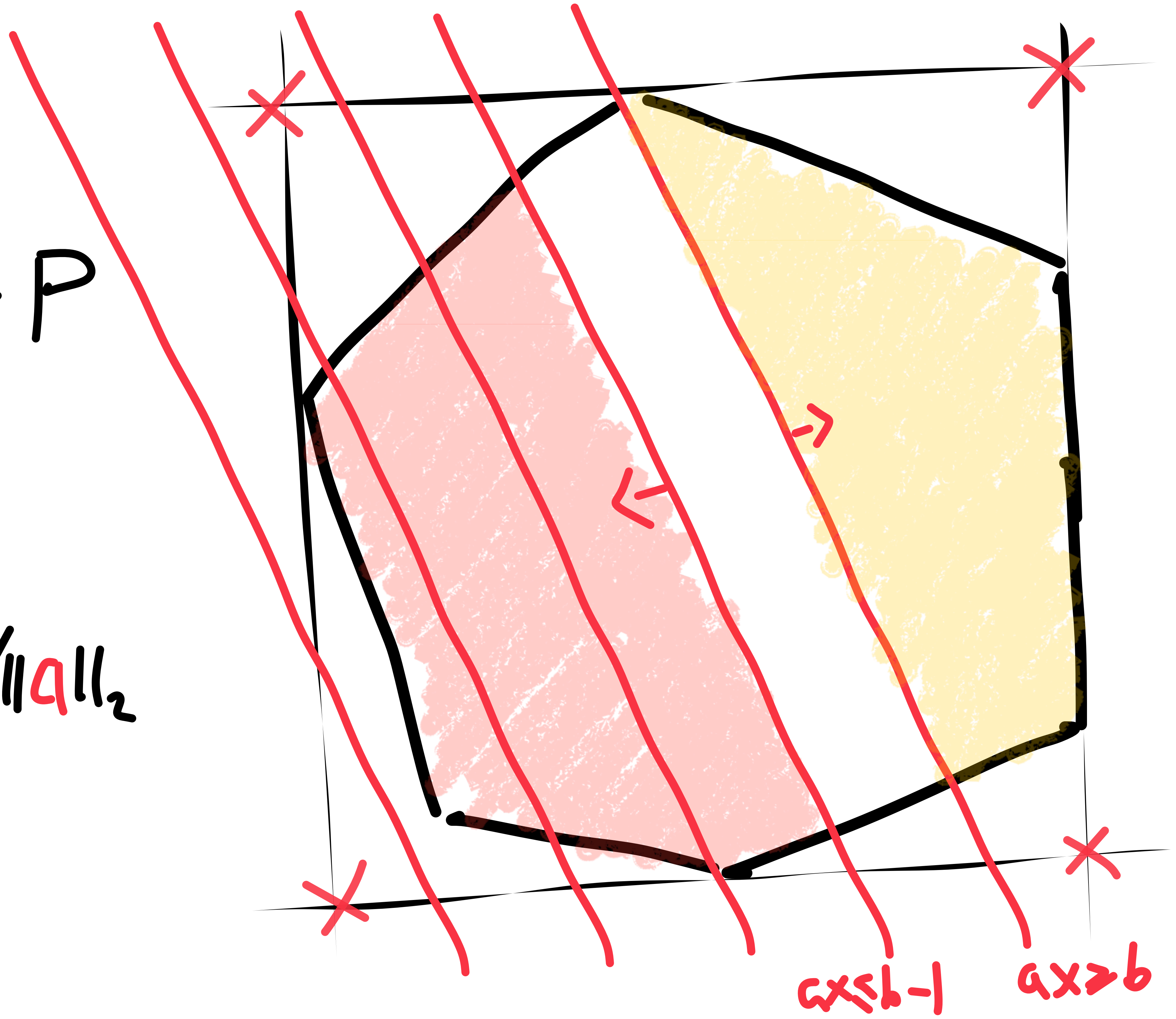


$*$ width of slab $= 1/\|a\|_2$



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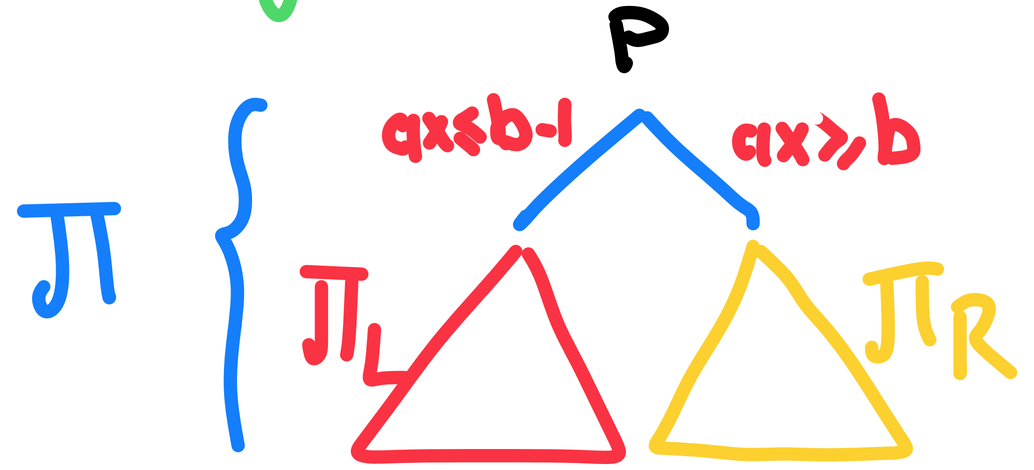

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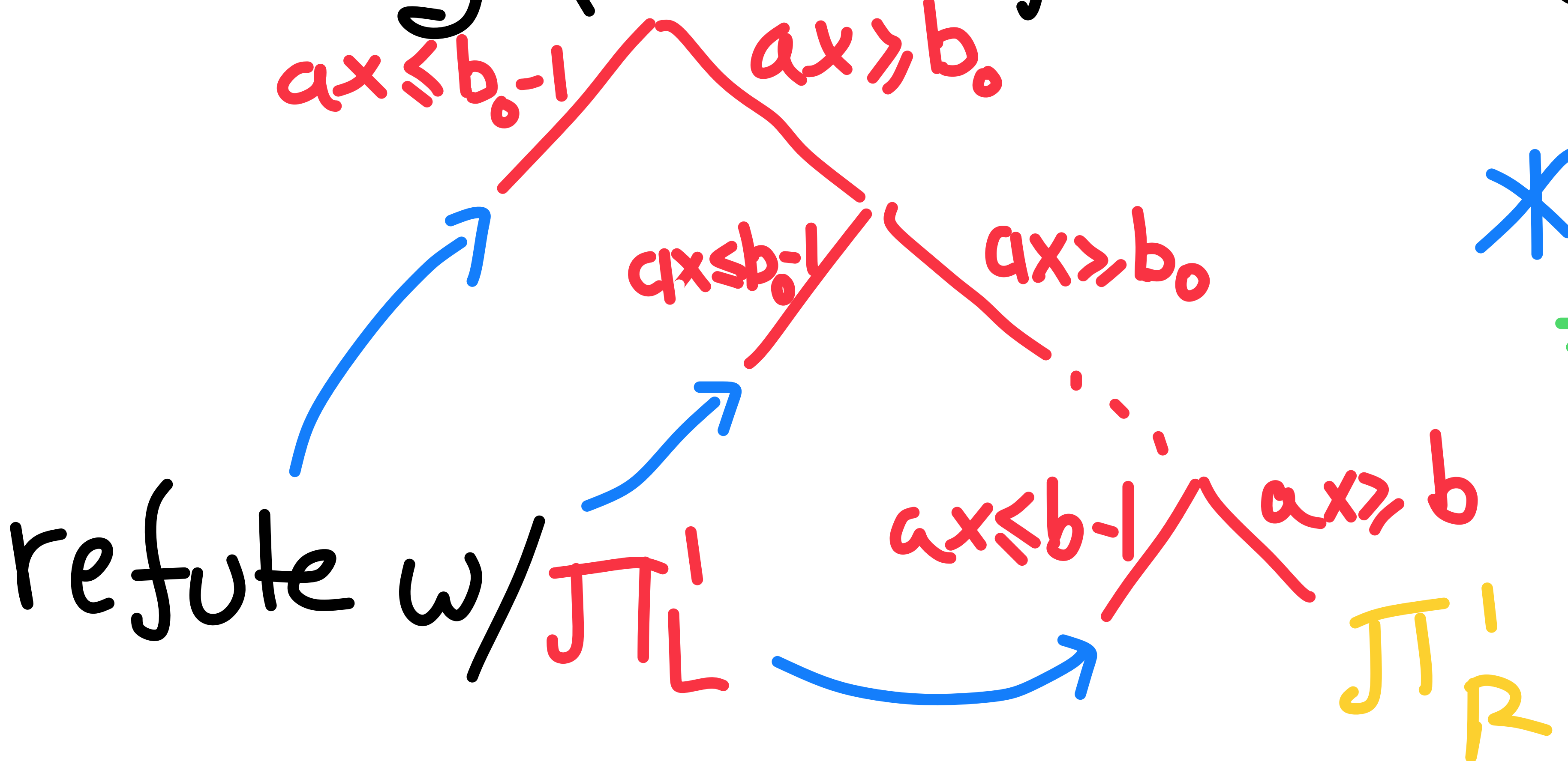
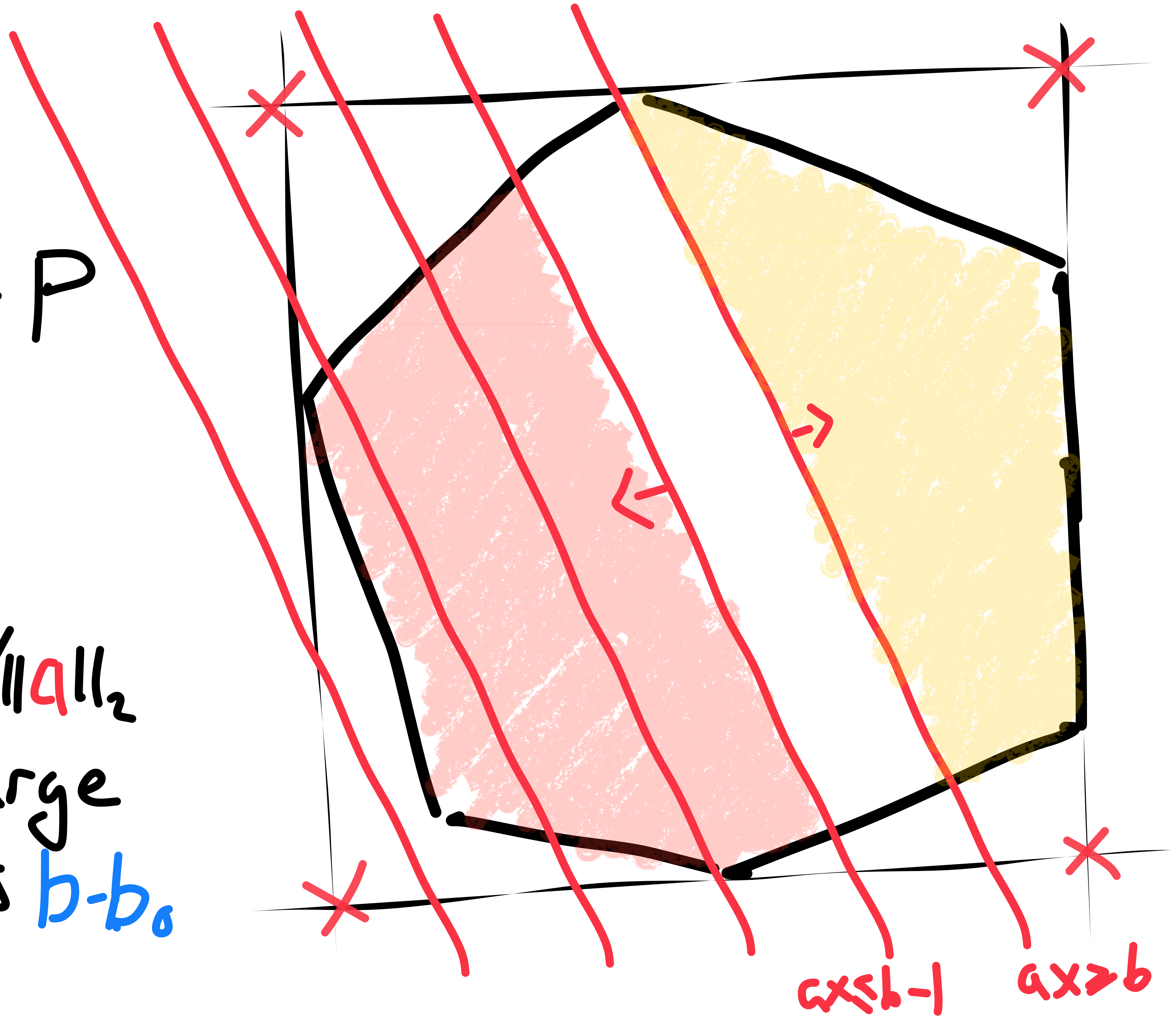


* width of slab $= 1/\|a\|_2$

$\Rightarrow$ if $\|a\|_2$ not large then neither is $b - b_0$



Stabbing Planes vs Cutting Planes

Thm: Every SP^* proof can be quasipolynomially translated into CP

Cor: $\exp(n^\varepsilon)$ lower bounds for SP^*

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Cor: CP has quasipolynomial size refutations of any unsat system of linear equations over a finite field

Refuting Systems of Equations in CP

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Step1. A general algorithm for refuting systems of equations

Refuting Systems of Equations in CP

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Step1. A general algorithm for refuting systems of equations
Step2. Implement in SP

Refuting Systems of Equations in CP

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Step 1. A general algorithm for refuting systems of equations
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Ex $\mathbb{F}_2$ linear equations

Refuting Systems of Equations

$$x_1 + x_2 + x_4 = 1$$
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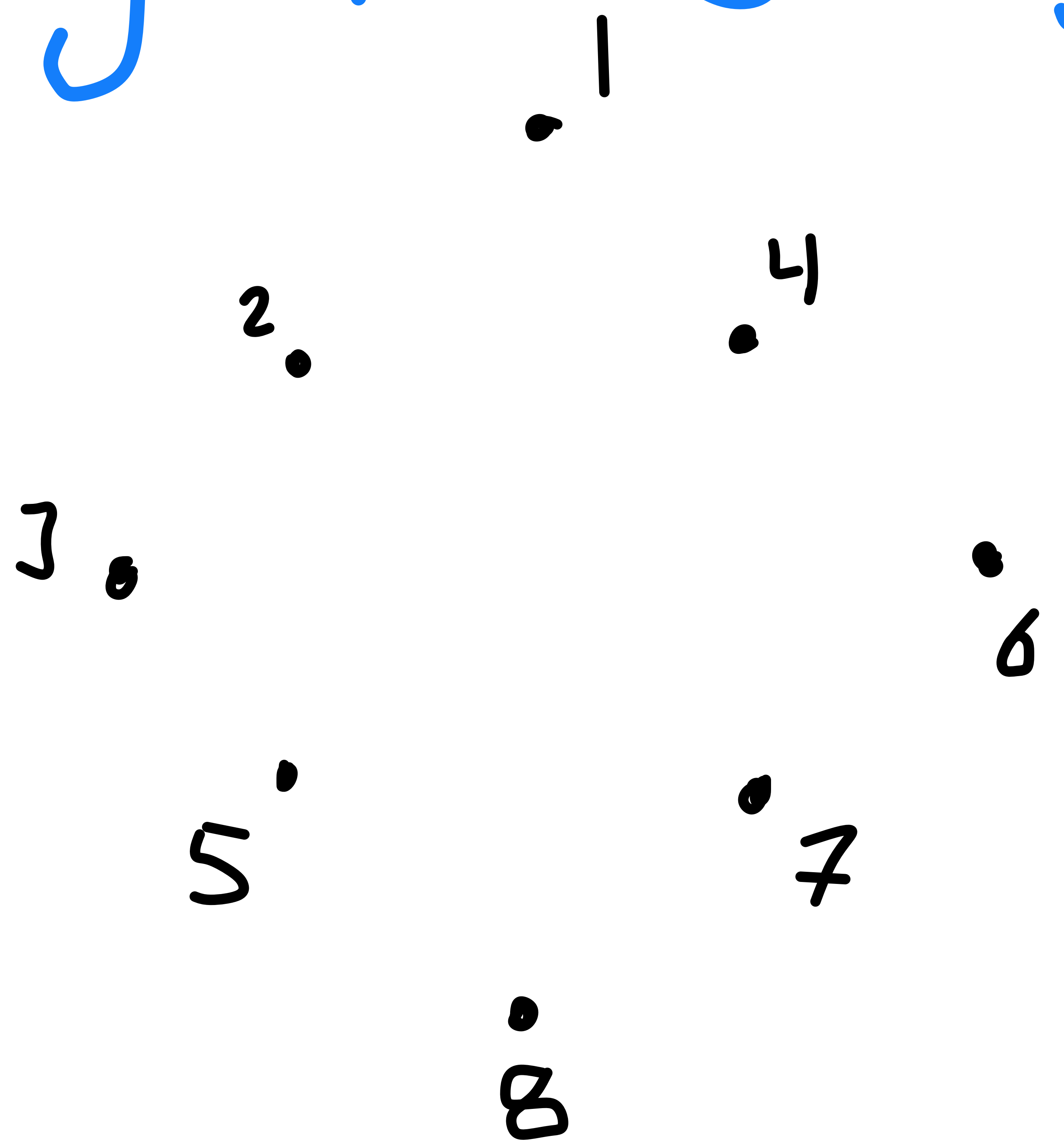

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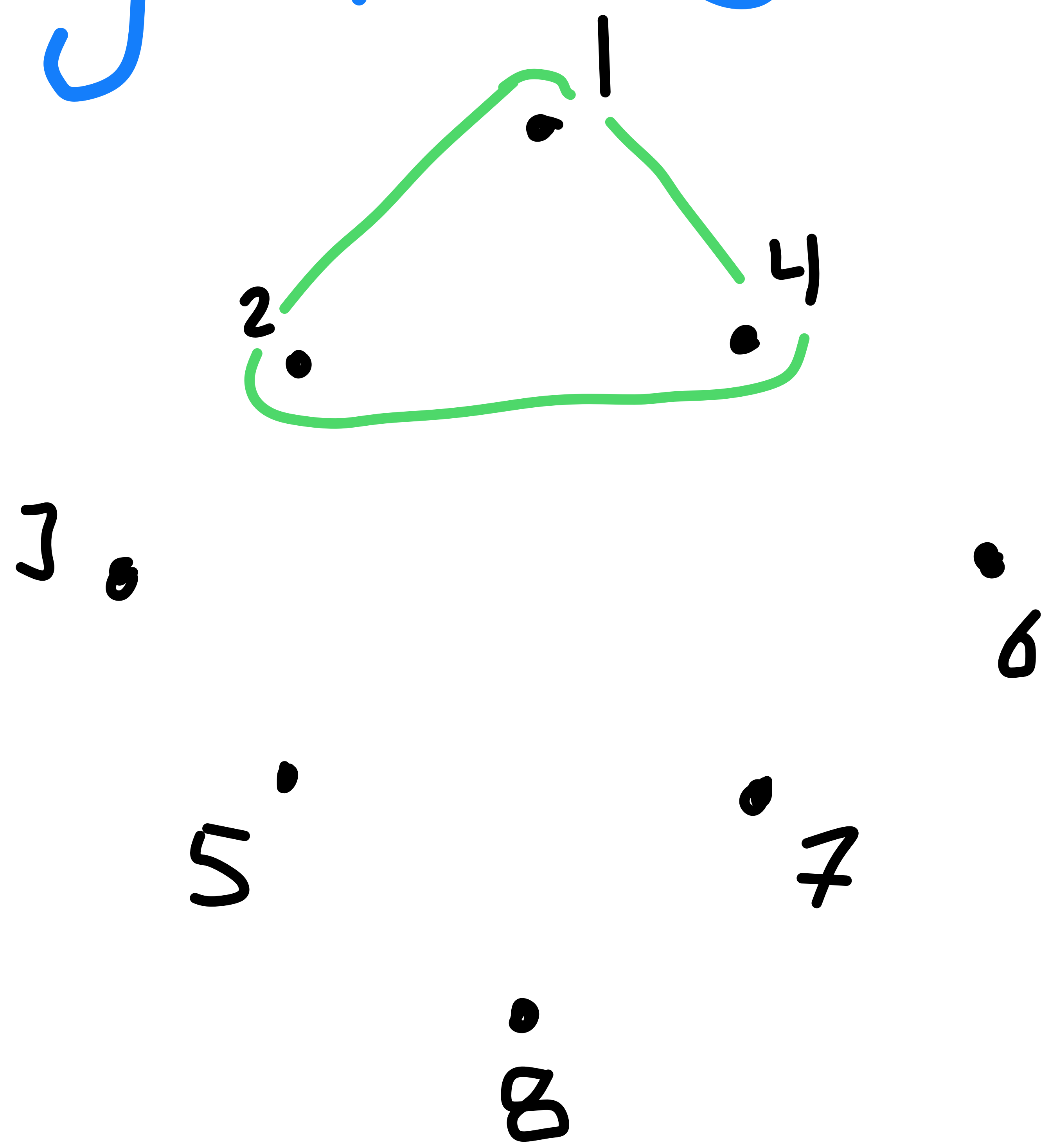

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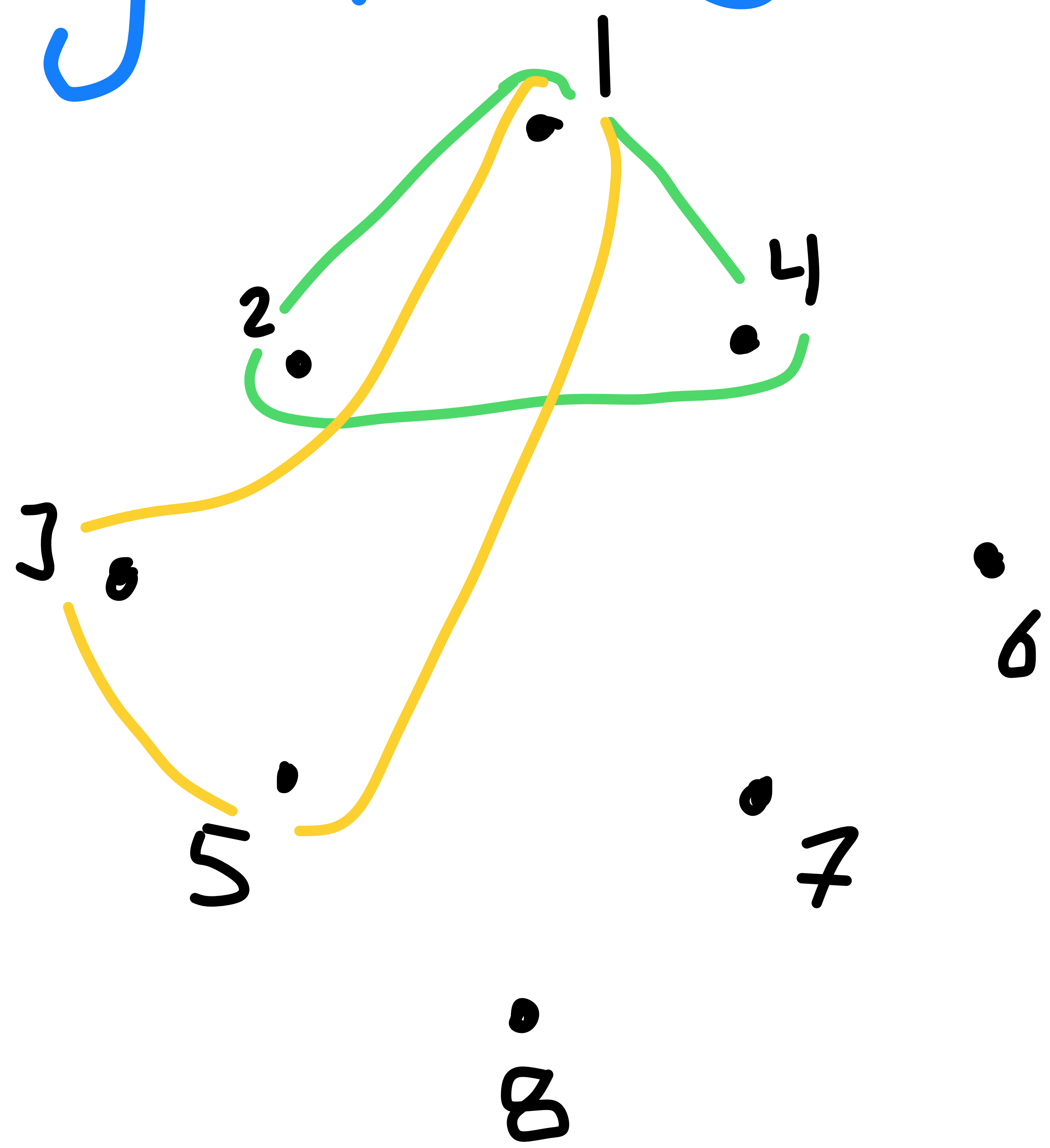

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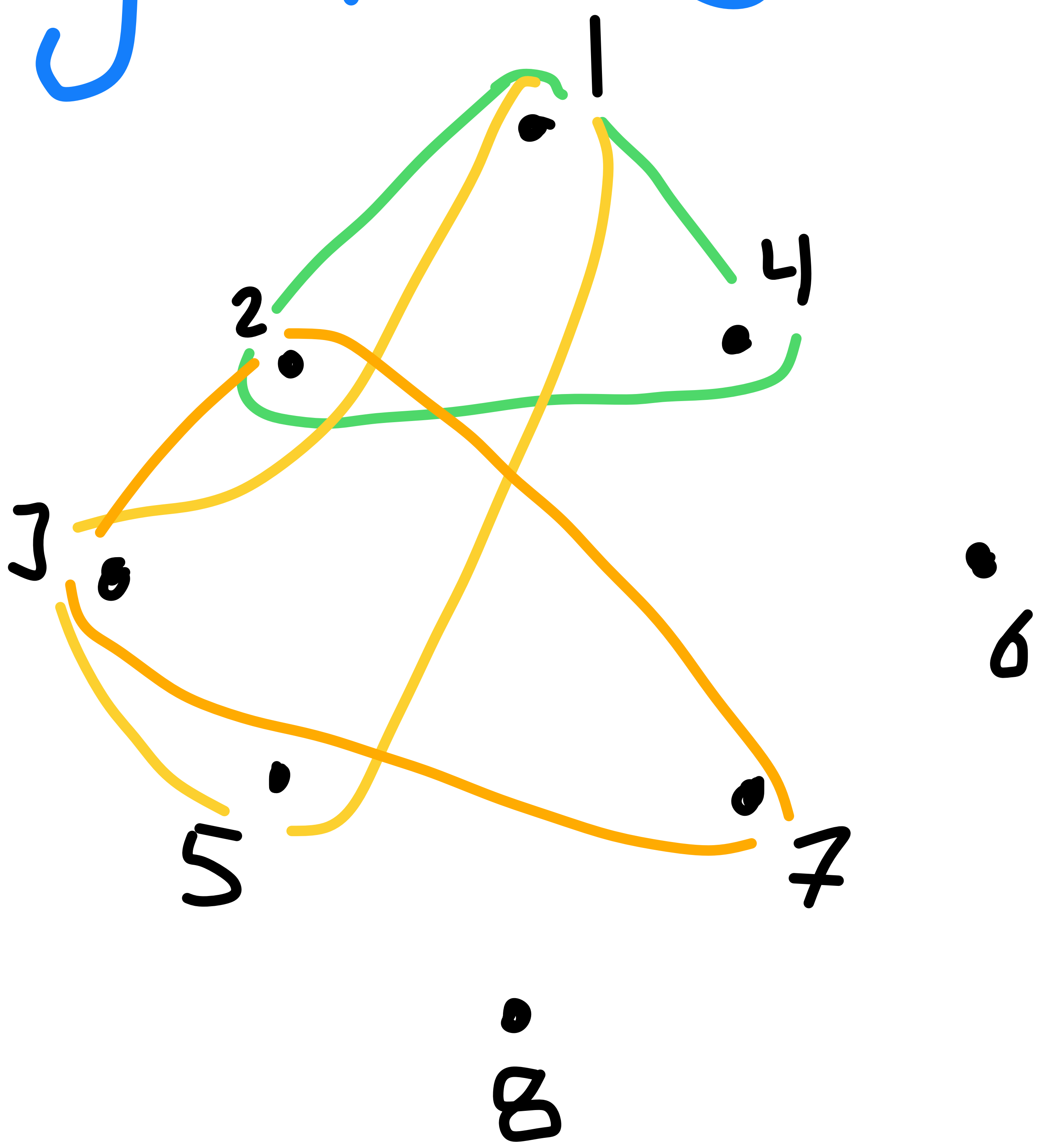

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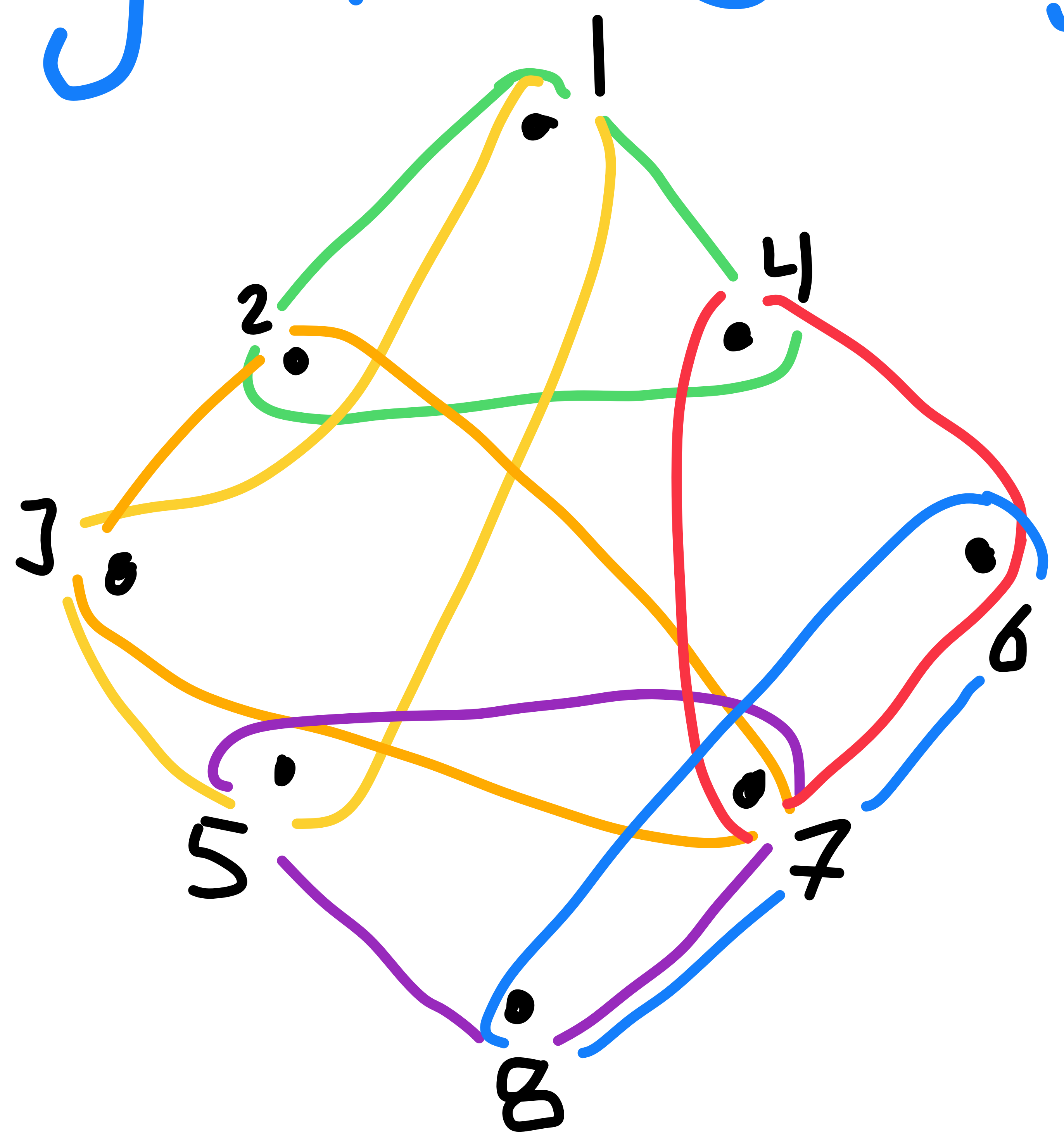

E = set of edges

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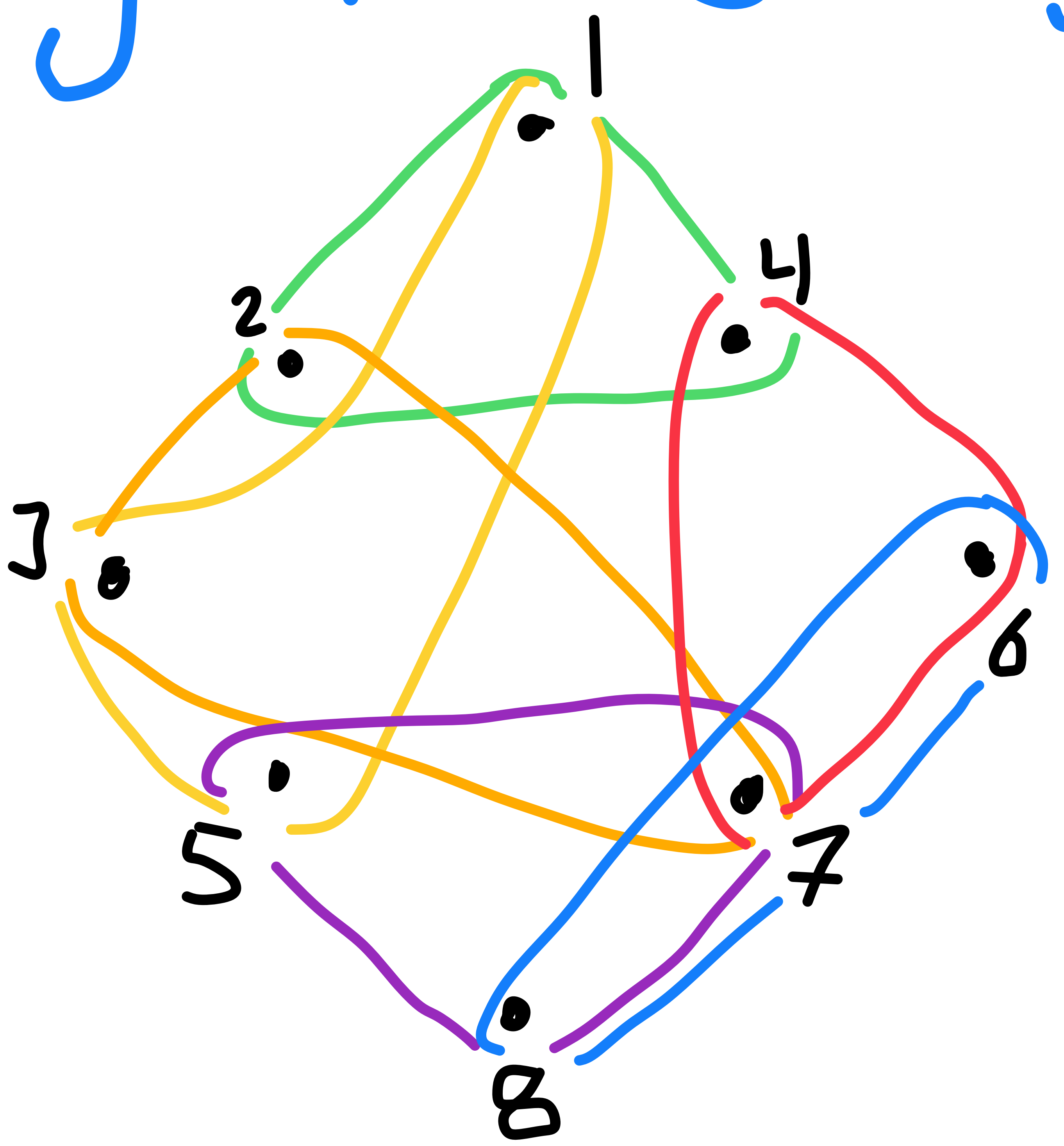

E = set of edges

1. Partition E into E_1, E_2 so that $|E_1| \approx |E_2|$

Refuting Systems of Equations

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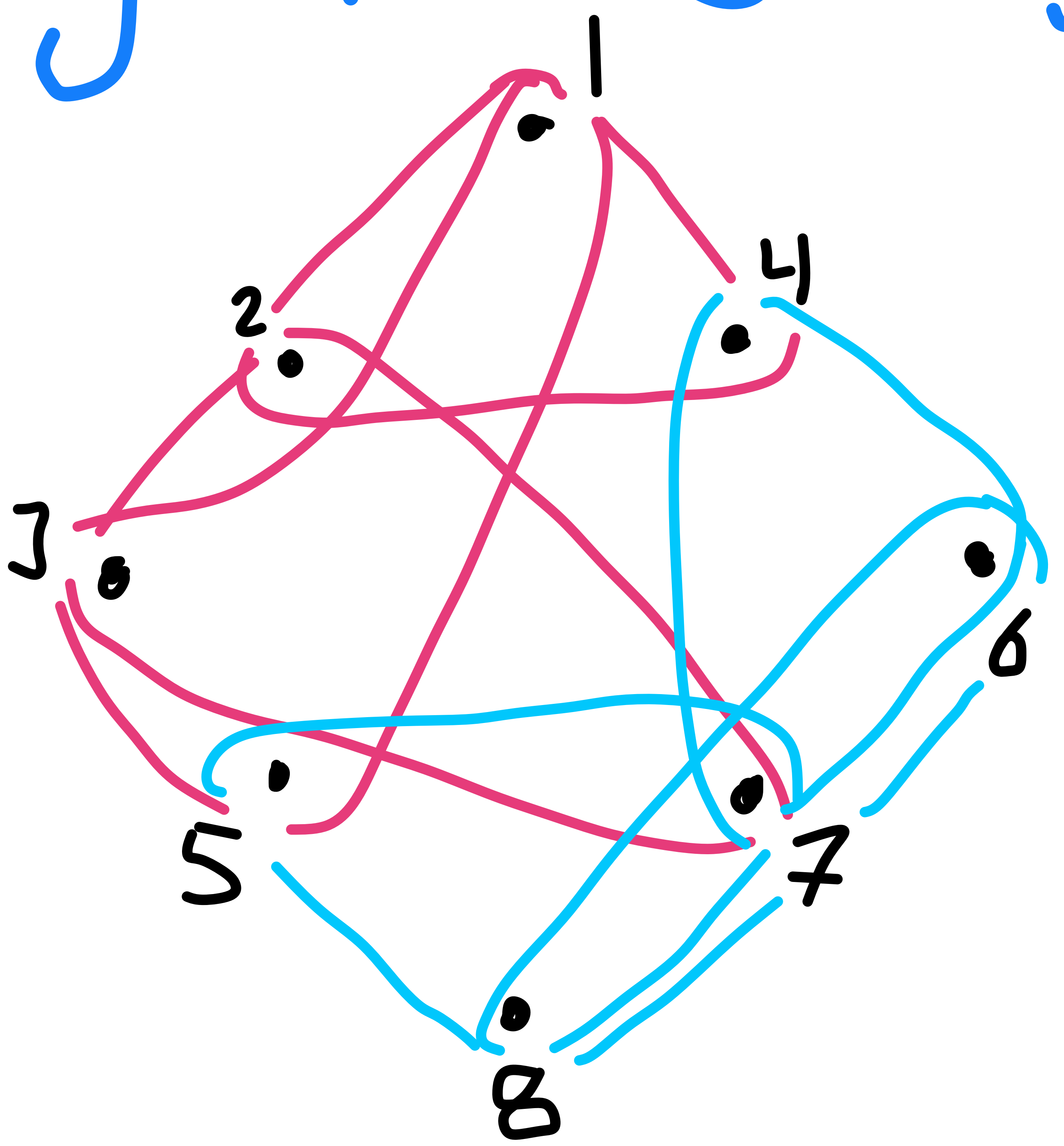

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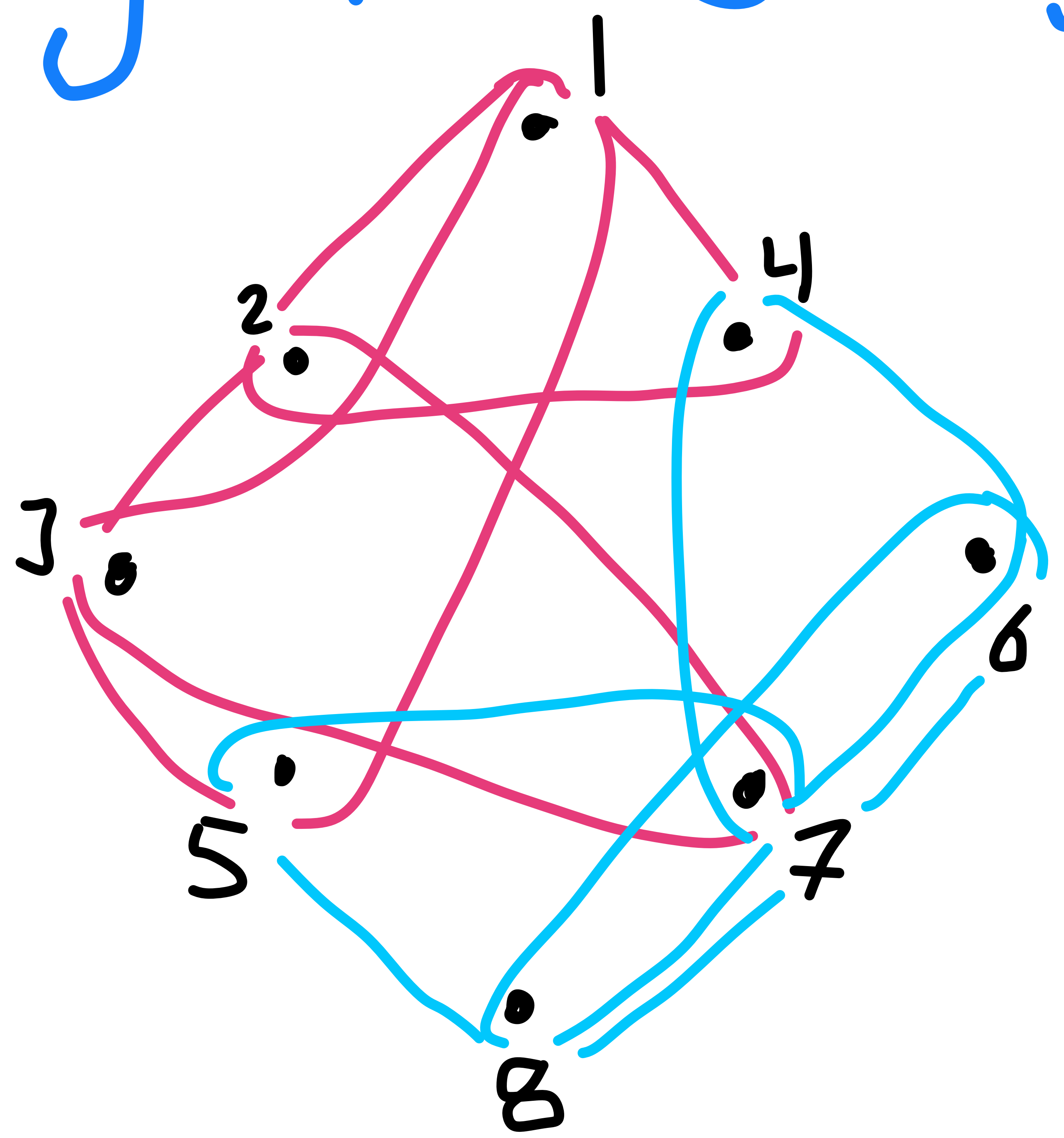

1. Partition E into E_1, E_2 so that $|E_1| \approx |E_2|$
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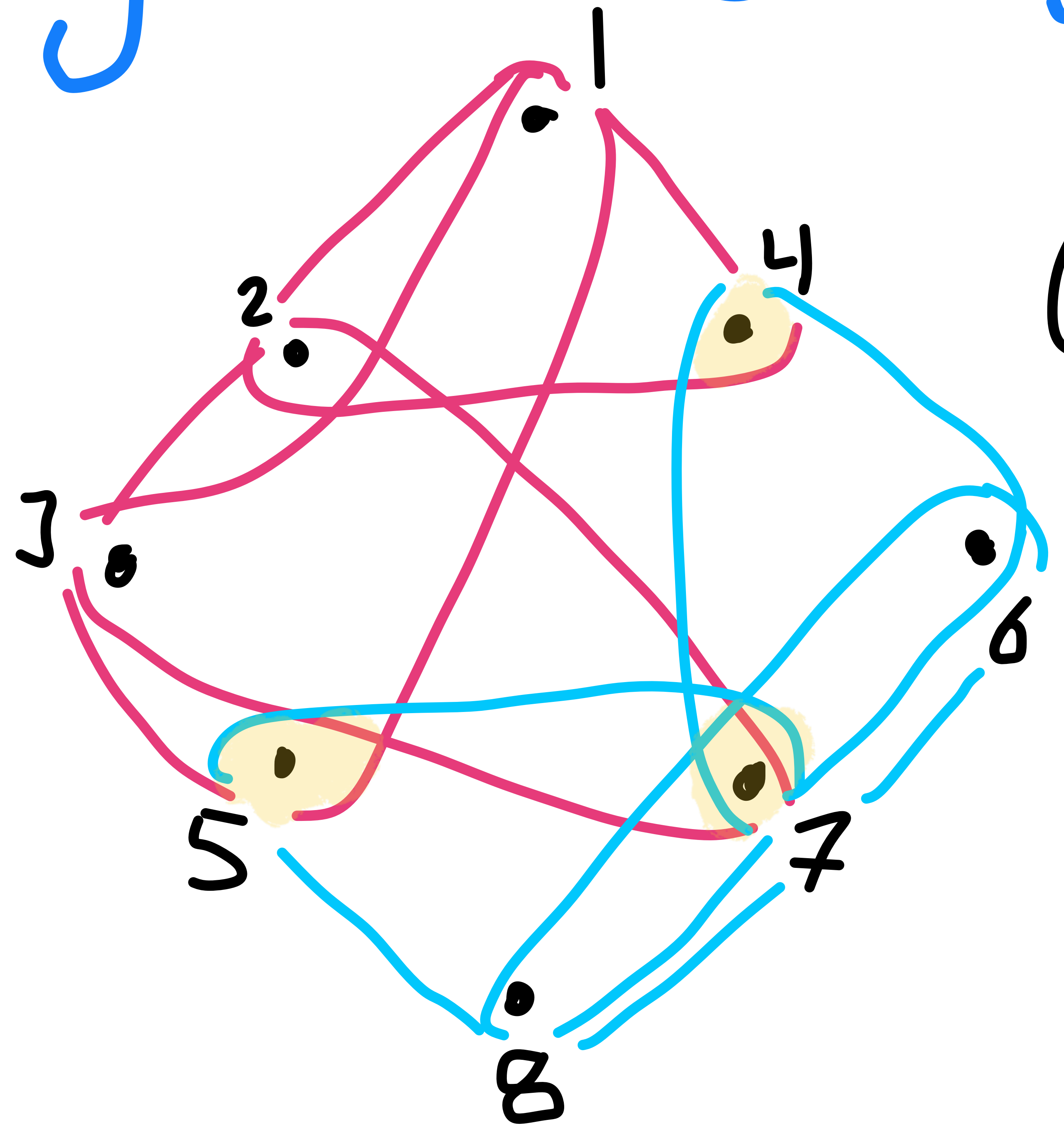

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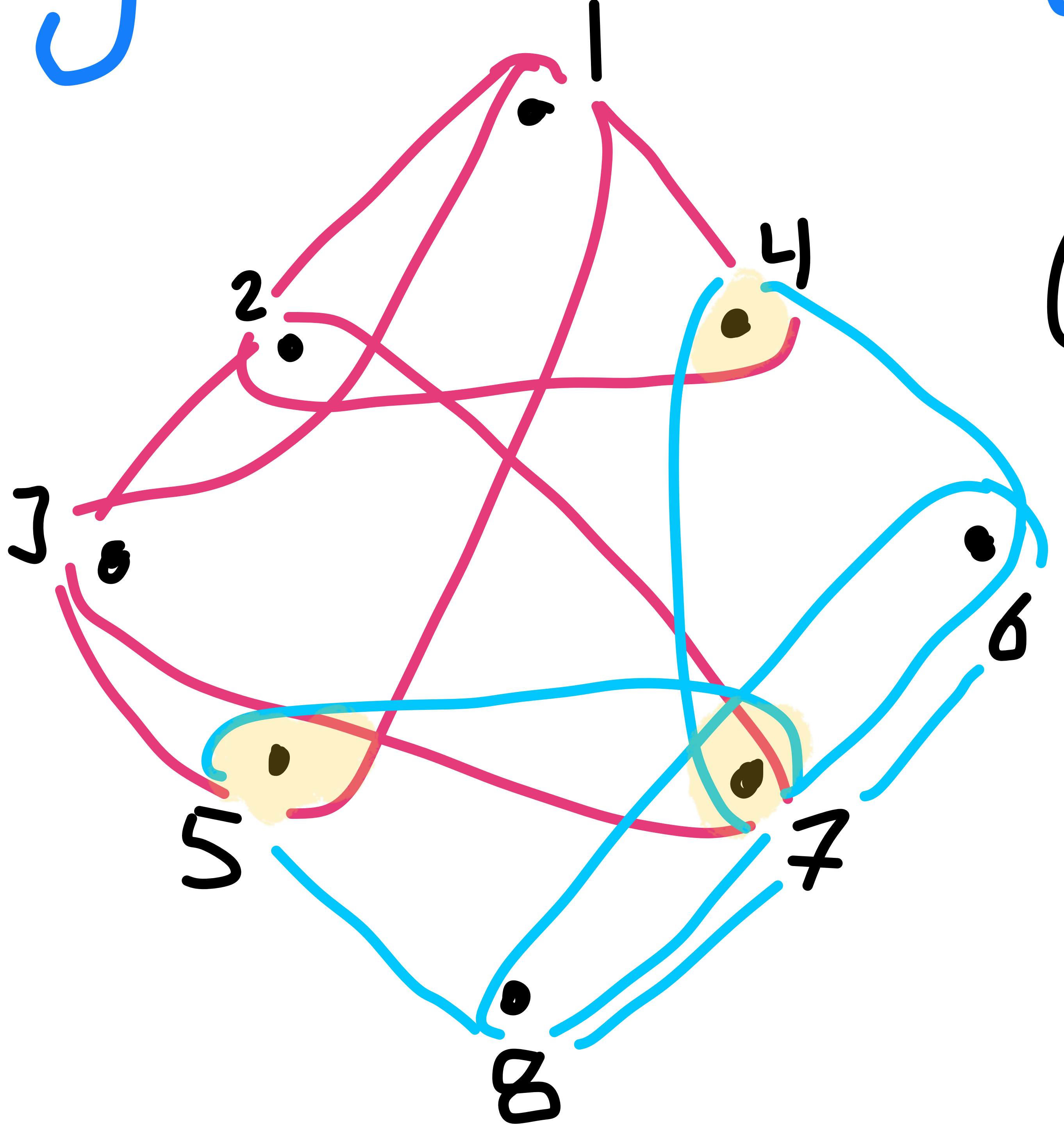

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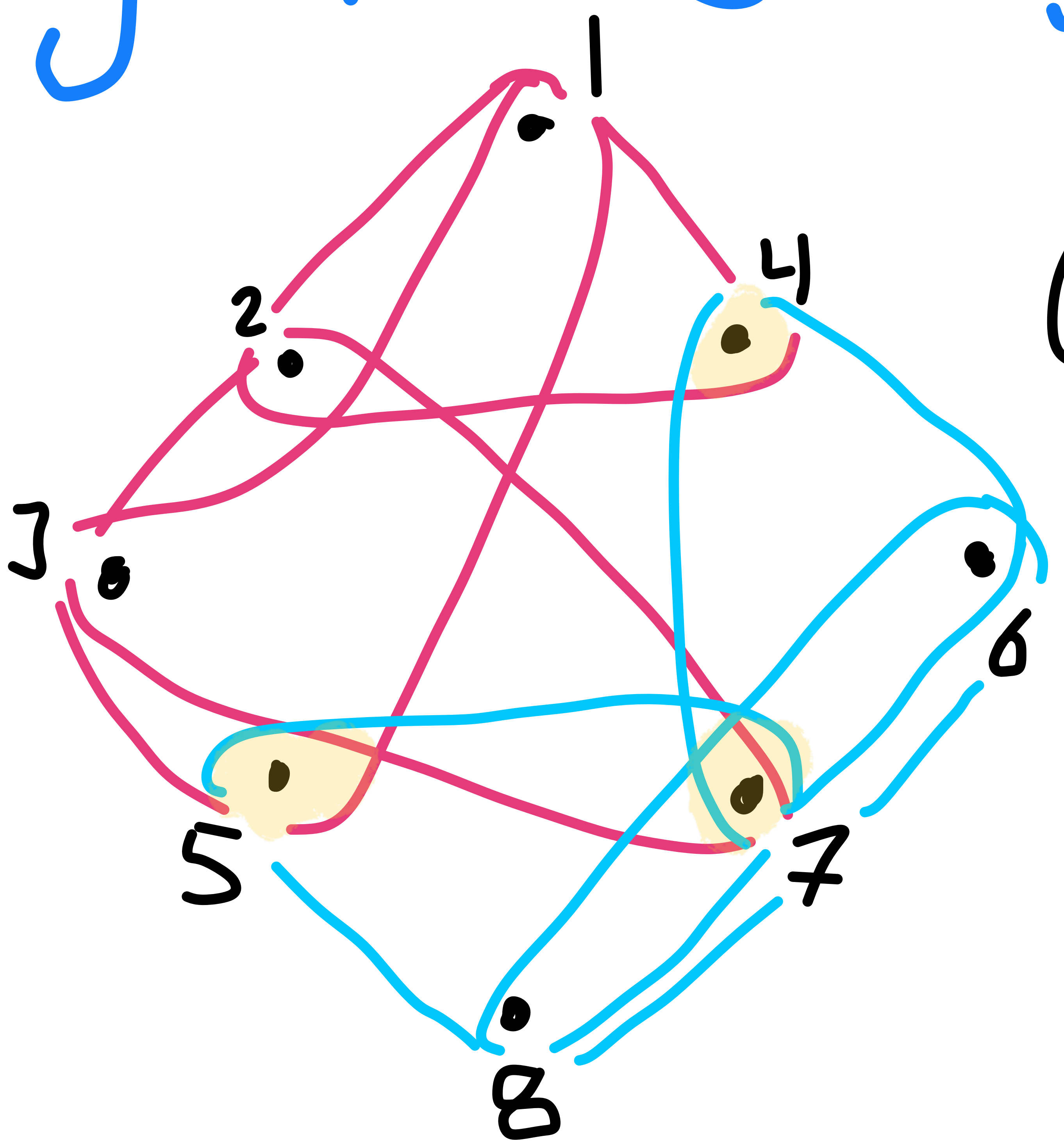

$Cut_2(E_1) = Cut_2(E_2) = \{x_4, x_5, x_7\}$

Suppose $x_4 + x_5 + x_7 \equiv 0 \mod 2$

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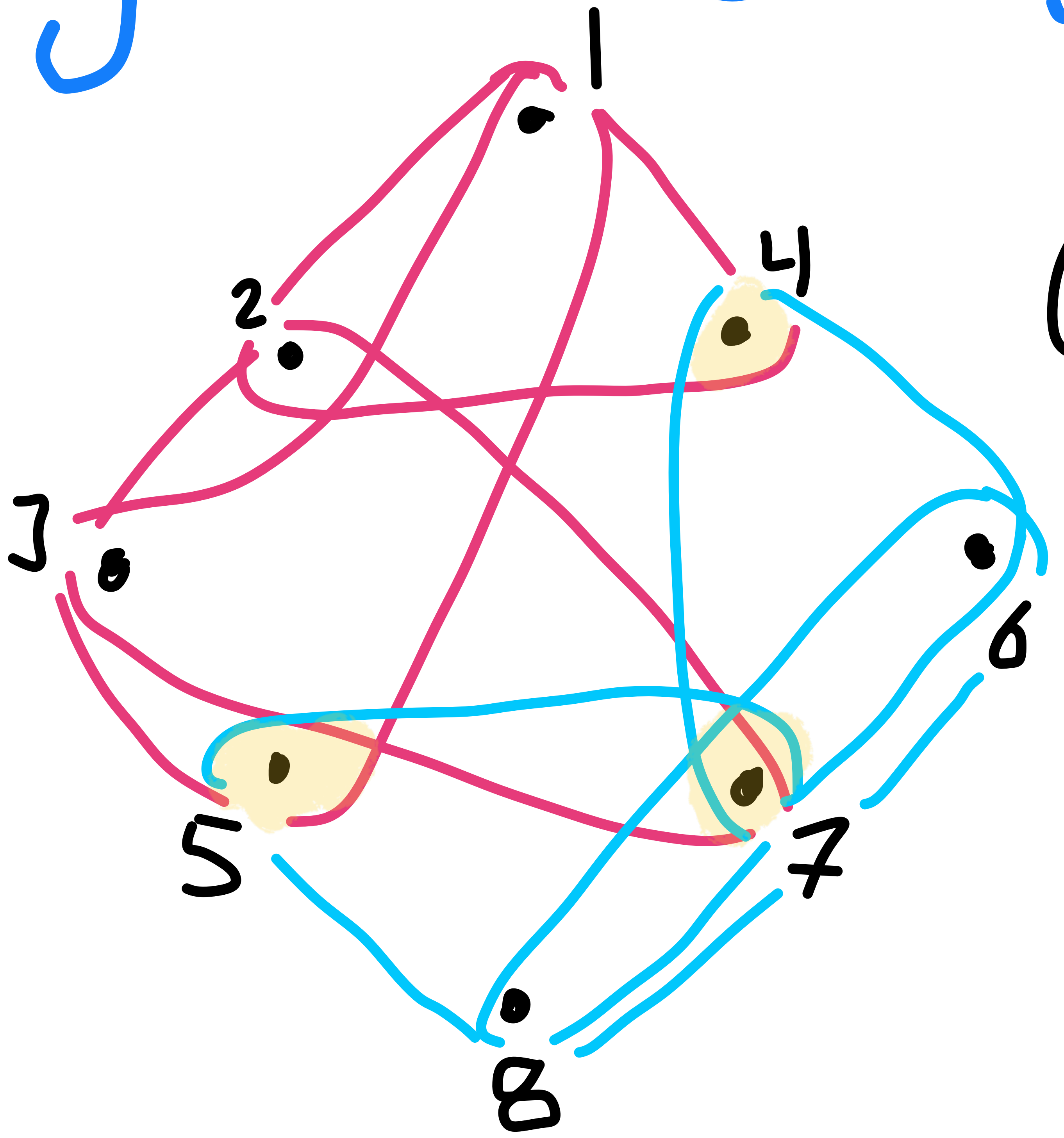

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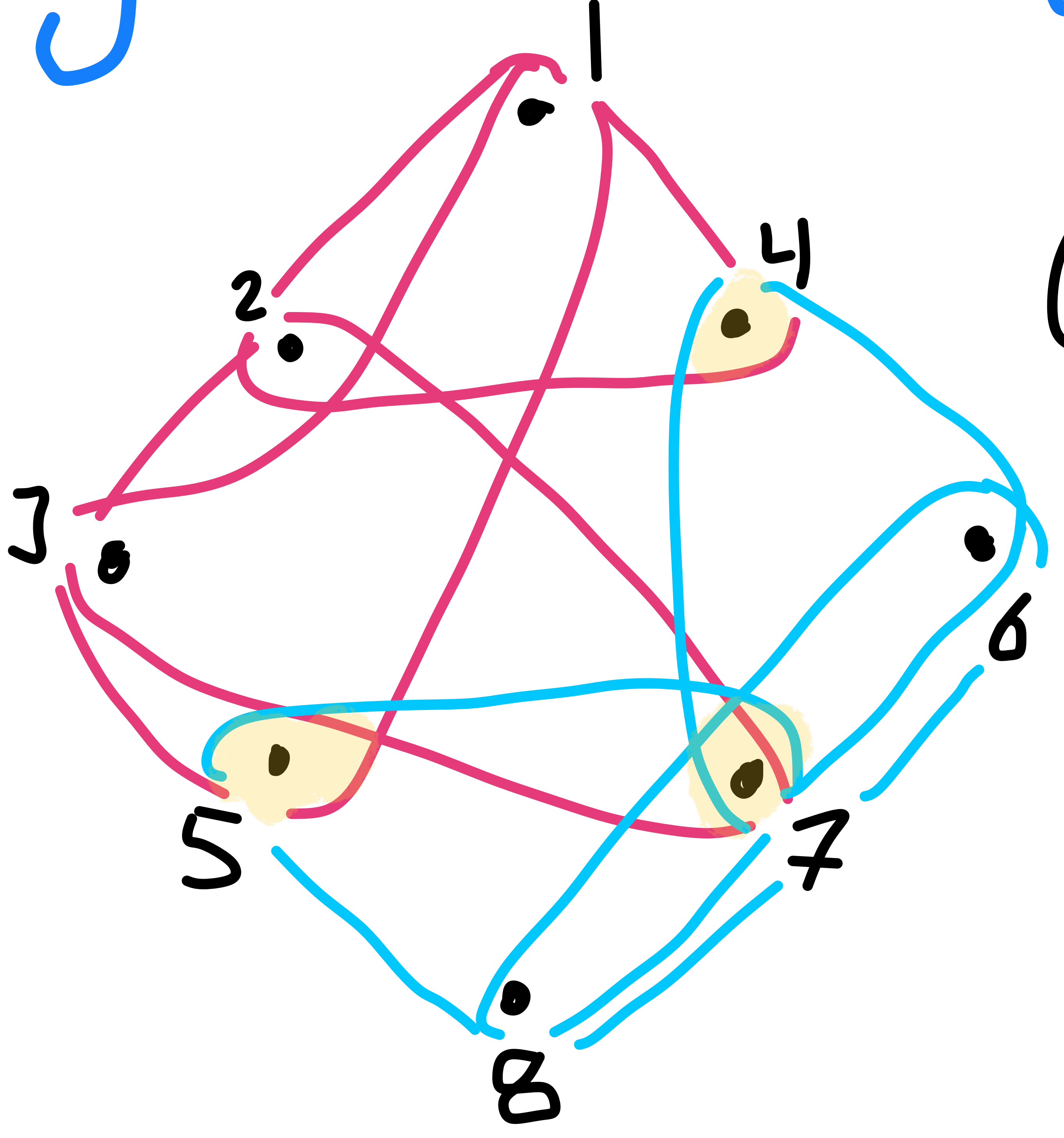

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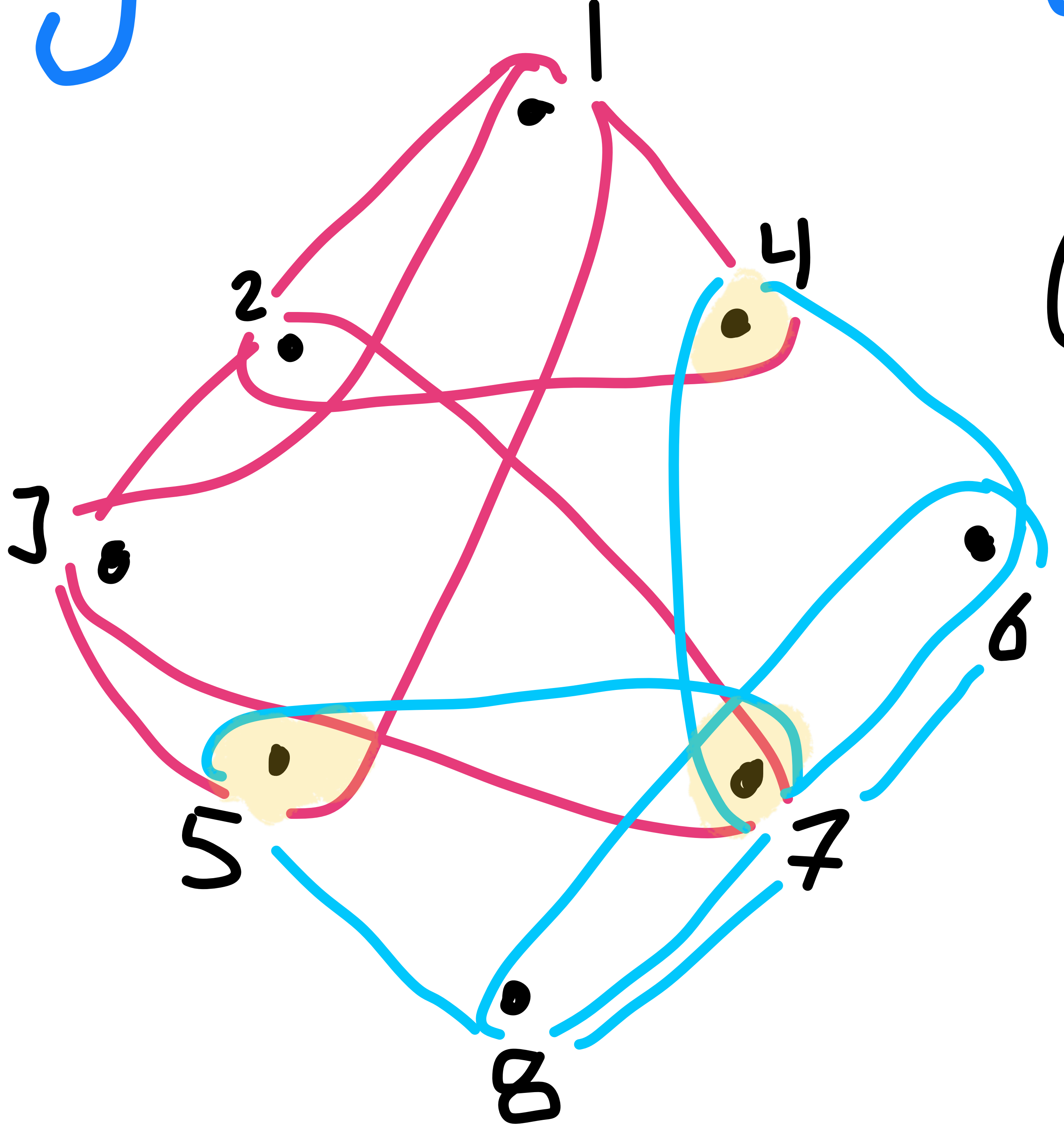

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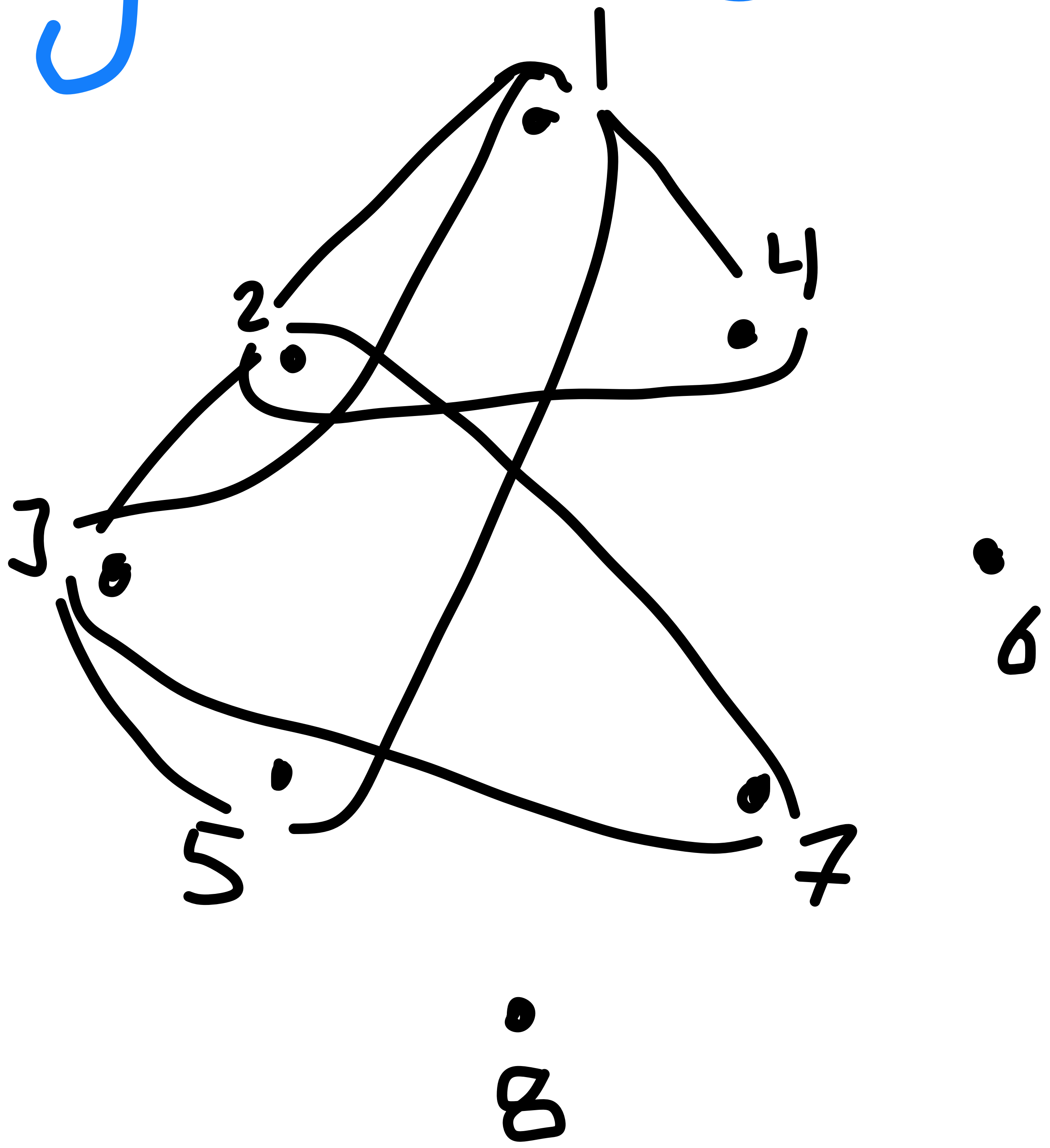

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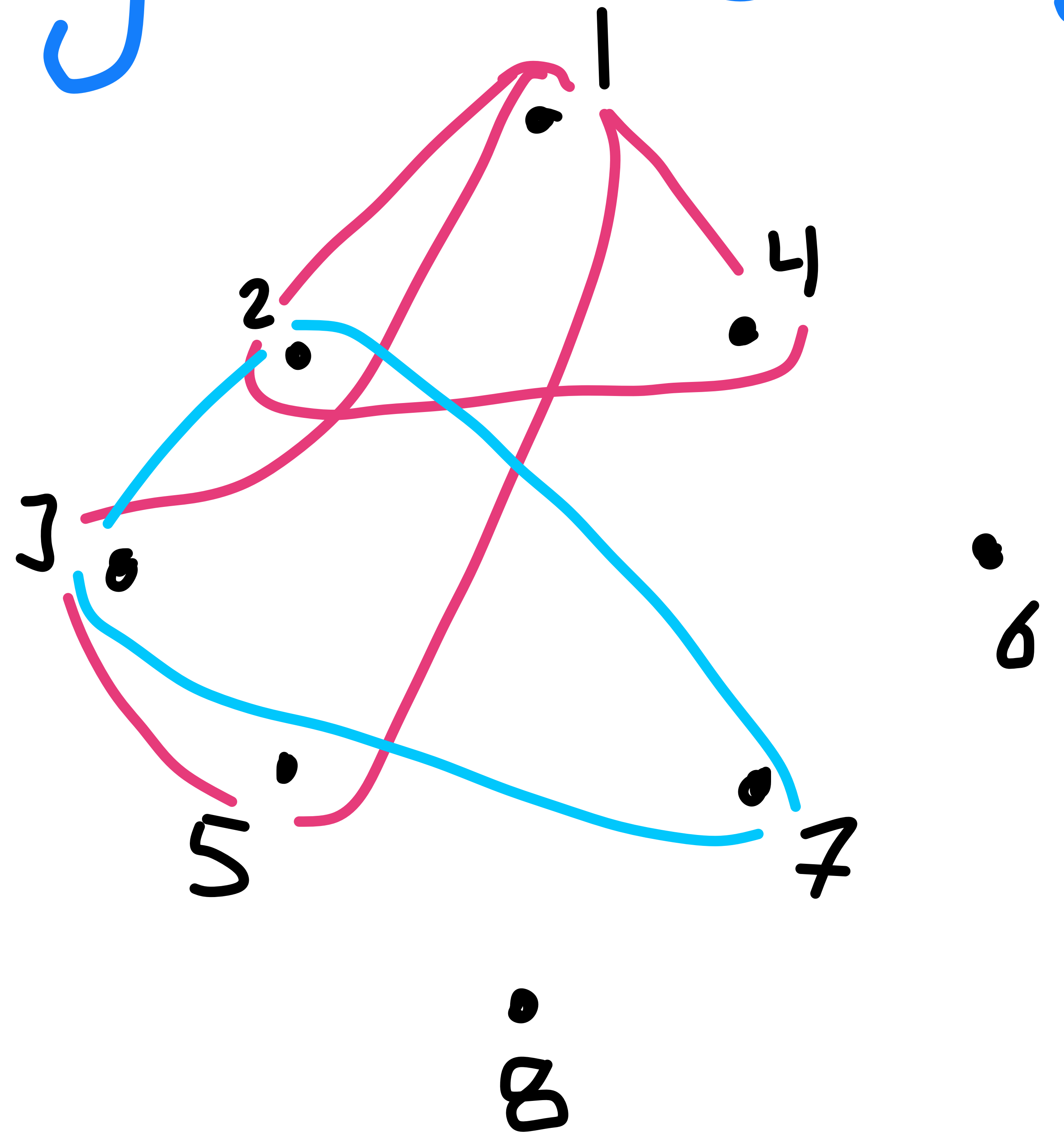

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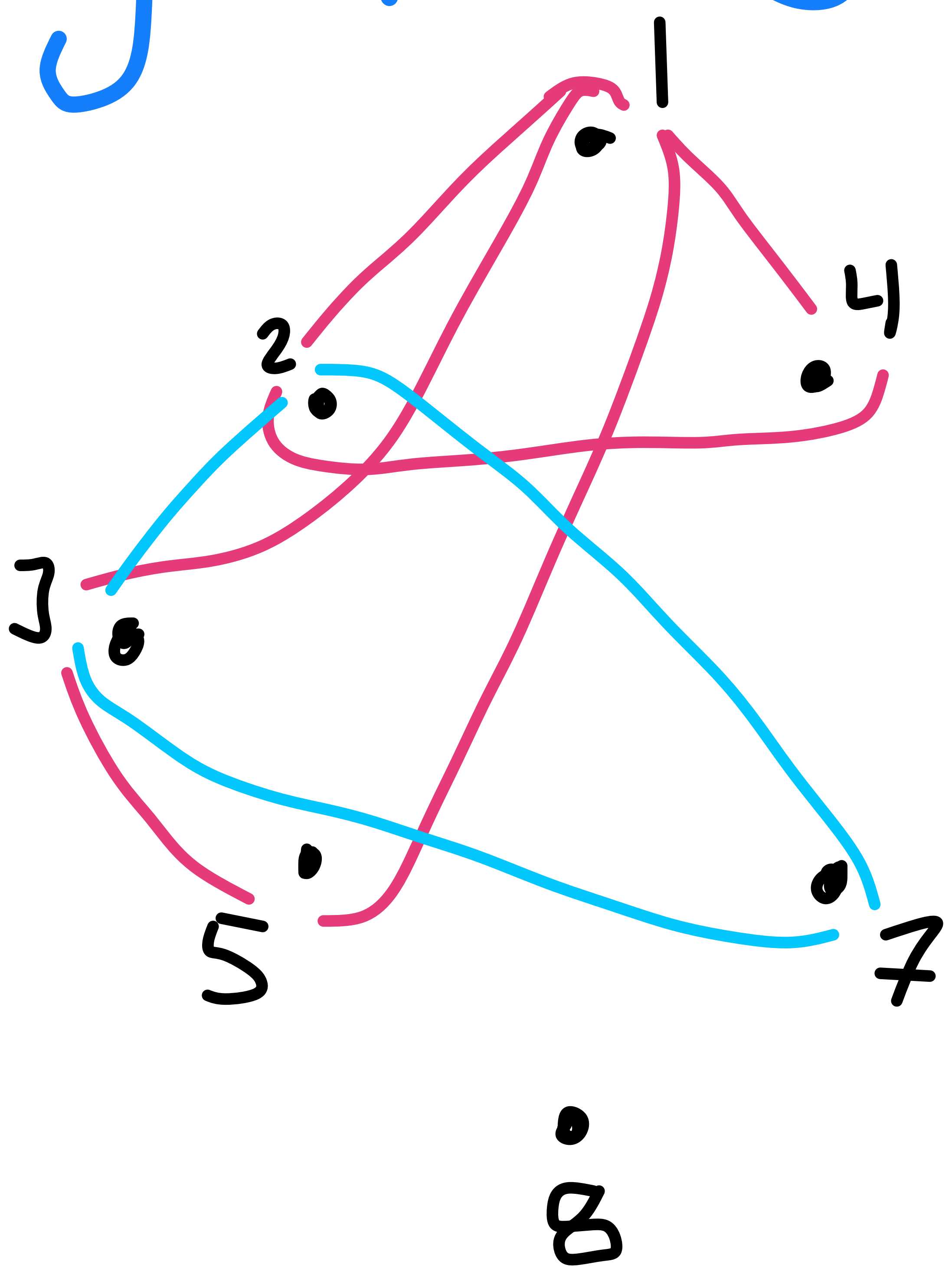

$Cut_2(E_1) = \{x_2, x_3, x_4, x_5\}$

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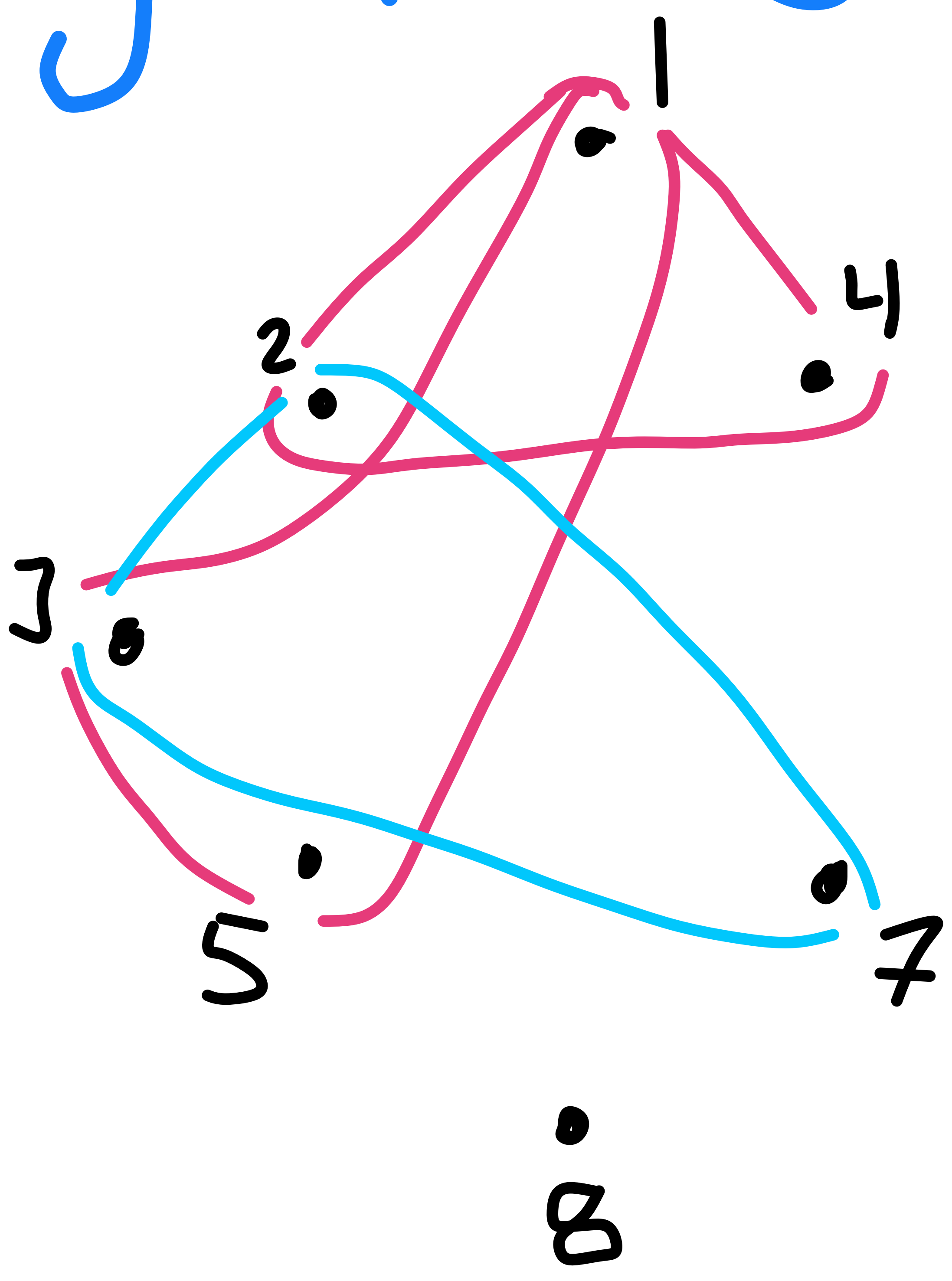

$Cut_2(E_1) = \{x_2, x_3, x_4, x_5\}$

Suppose $x_2 + x_3 + x_4 + x_5 \equiv 1 \bmod 2$

$\therefore$ $Cut_2(E_2) = \{x_2, x_3, x_7\}$

Suppose $x_2 + x_3 + x_7 \equiv 1 \bmod 2$

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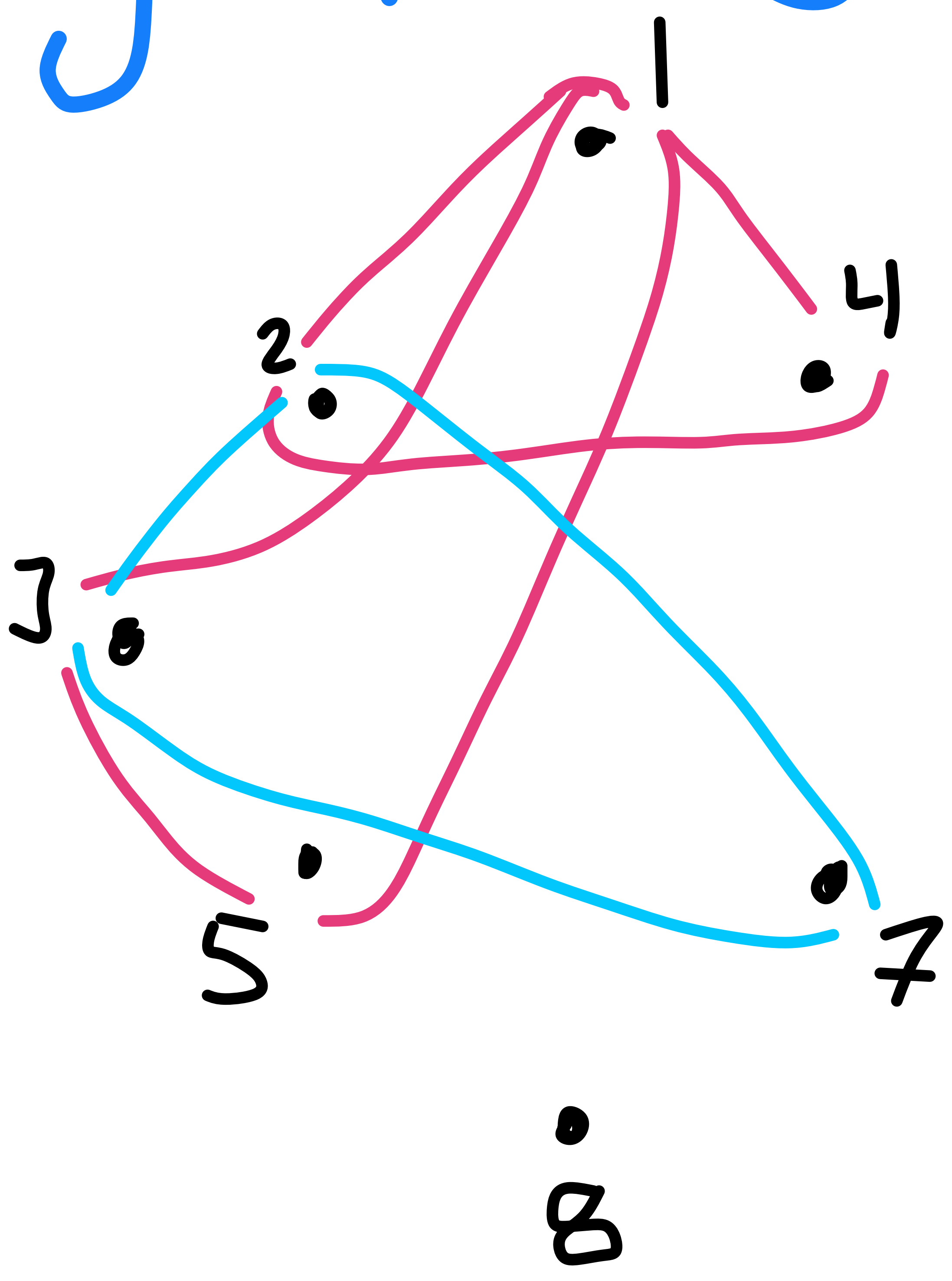

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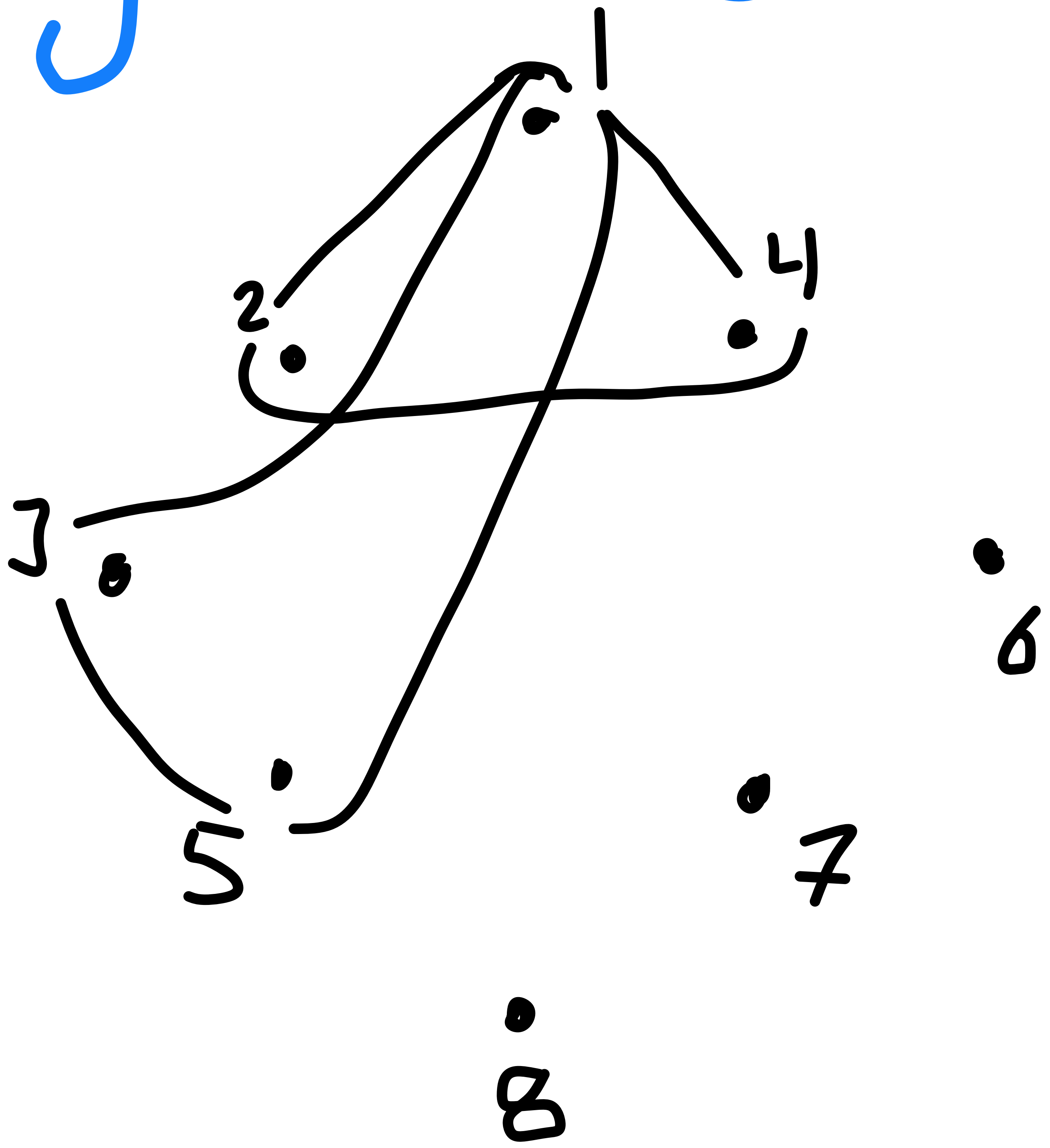

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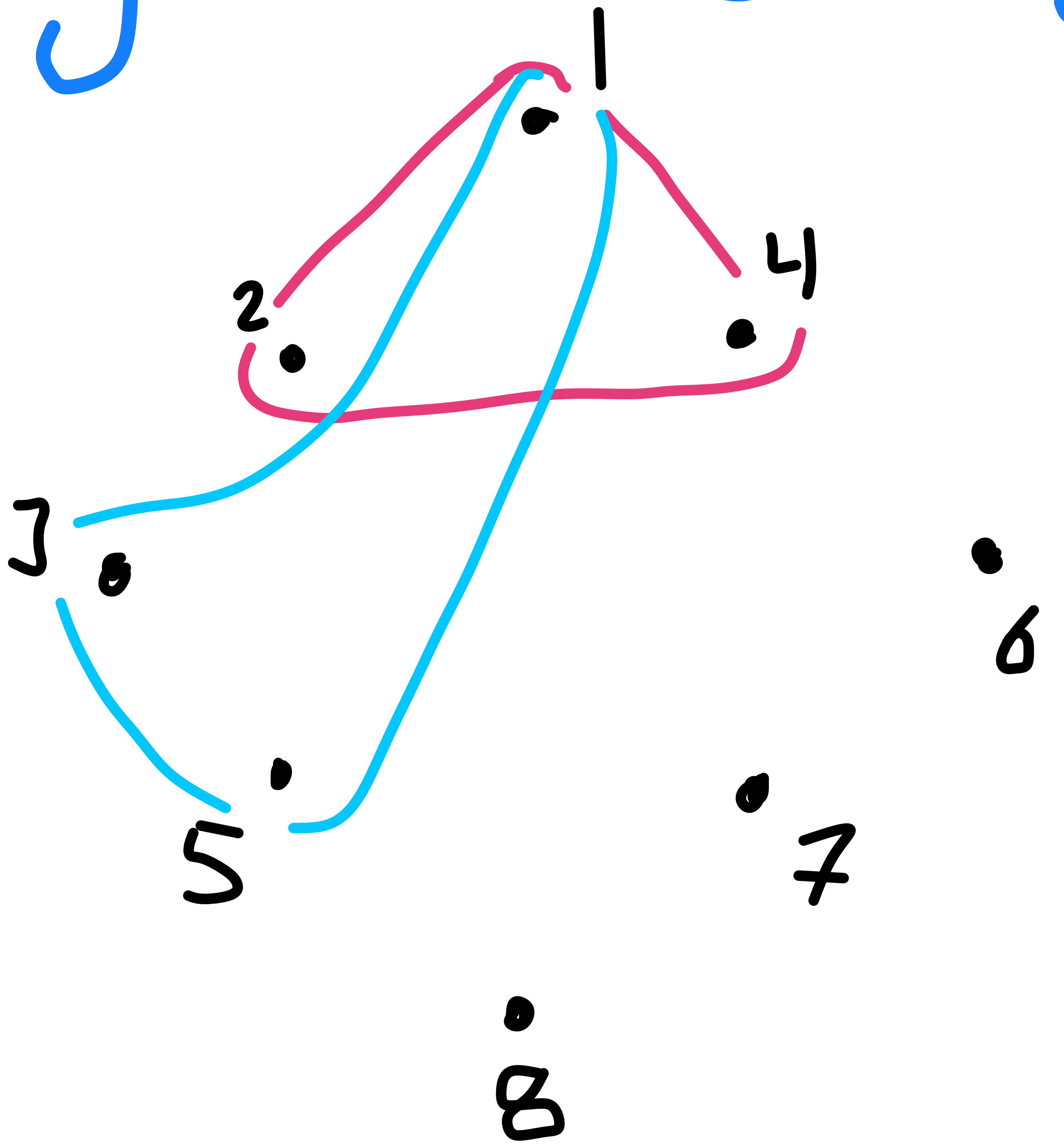

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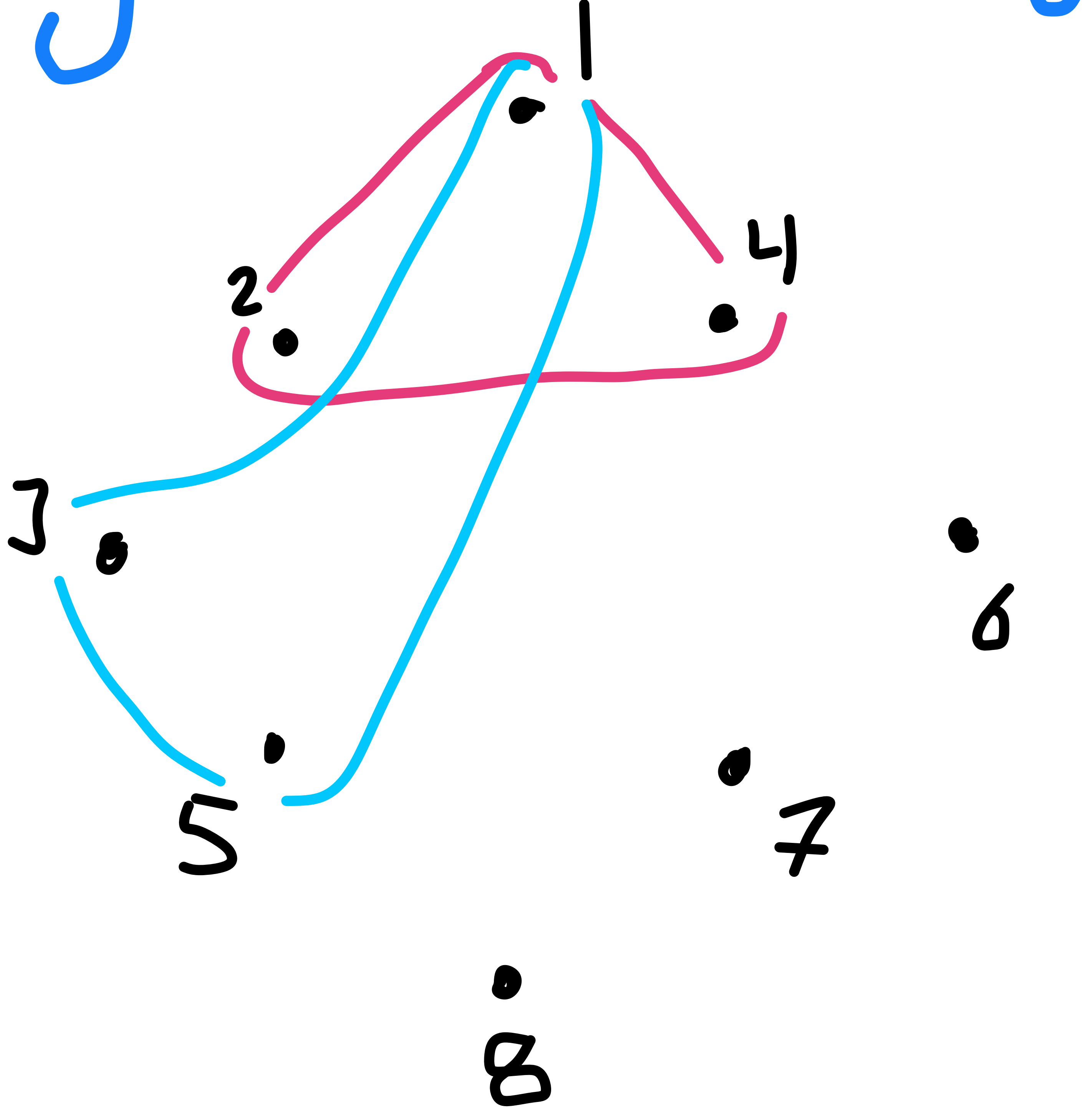

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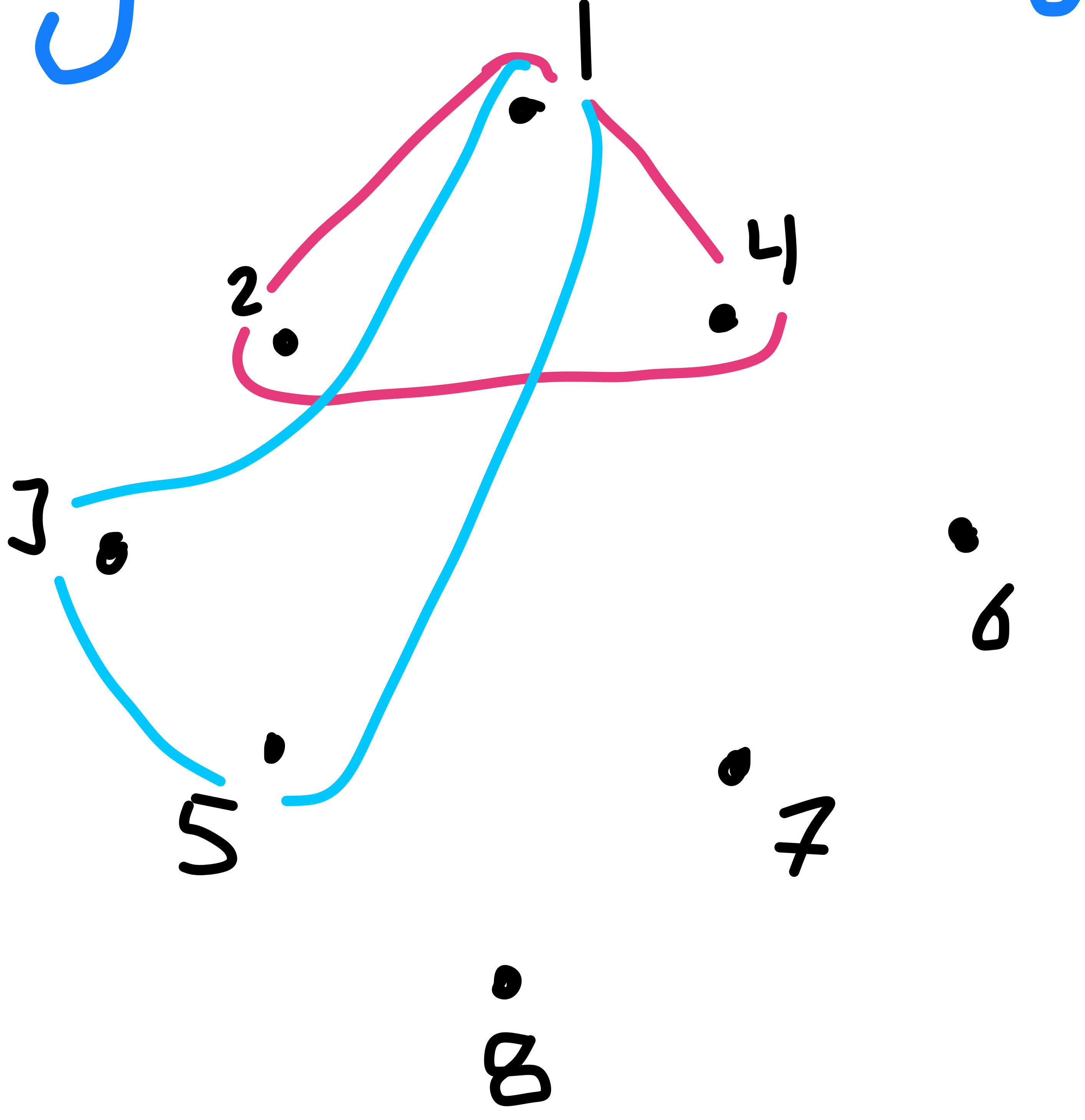

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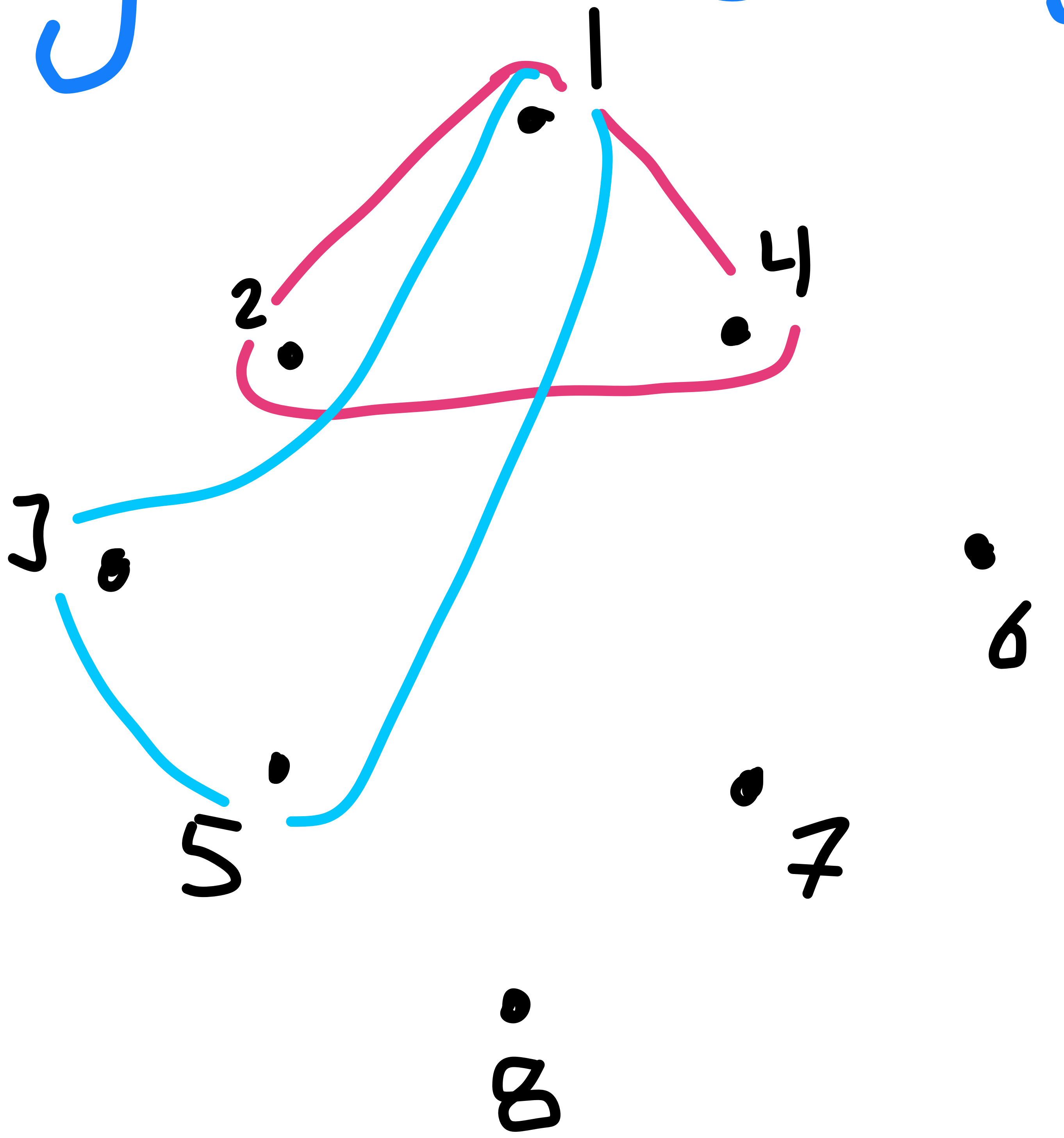

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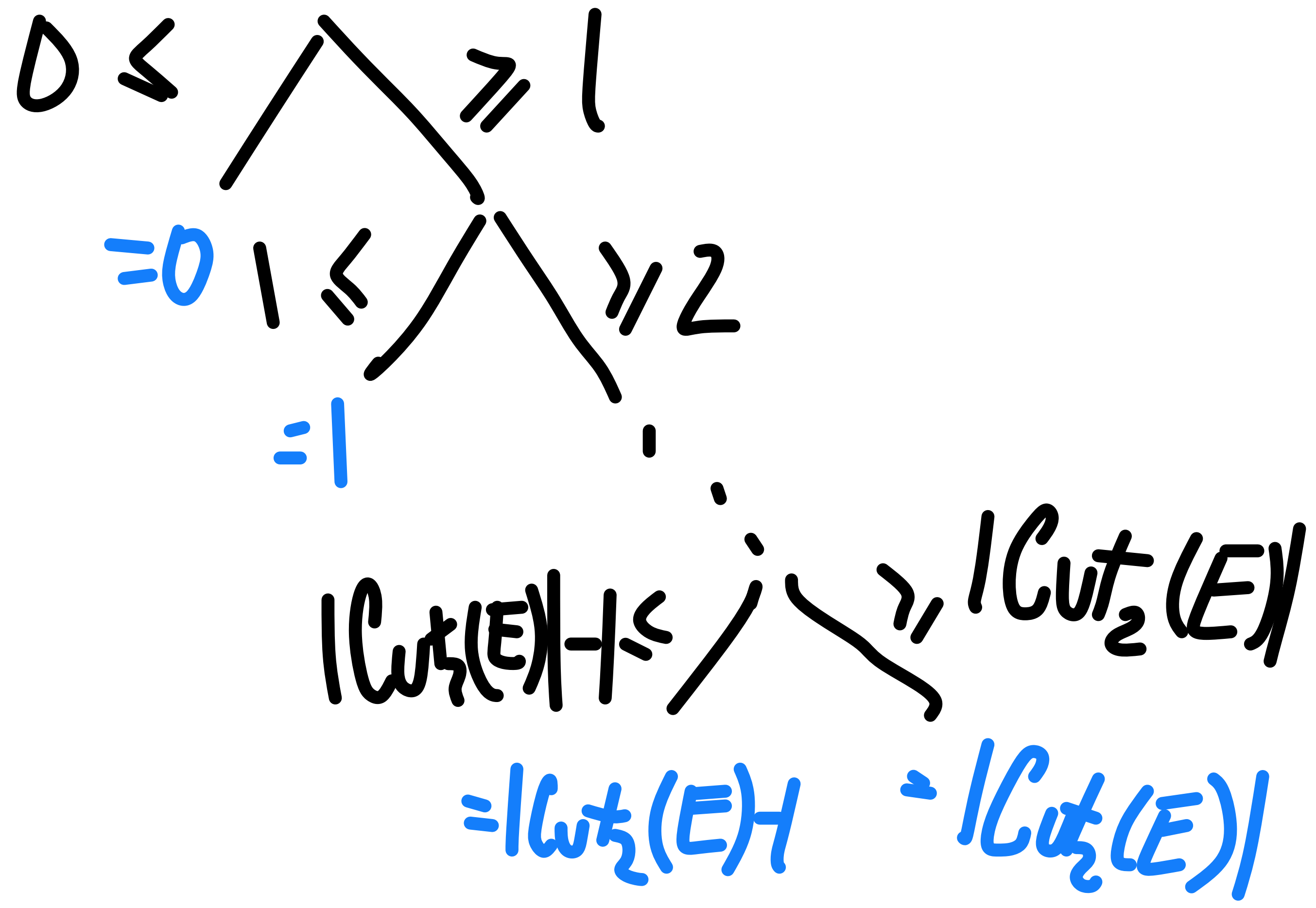

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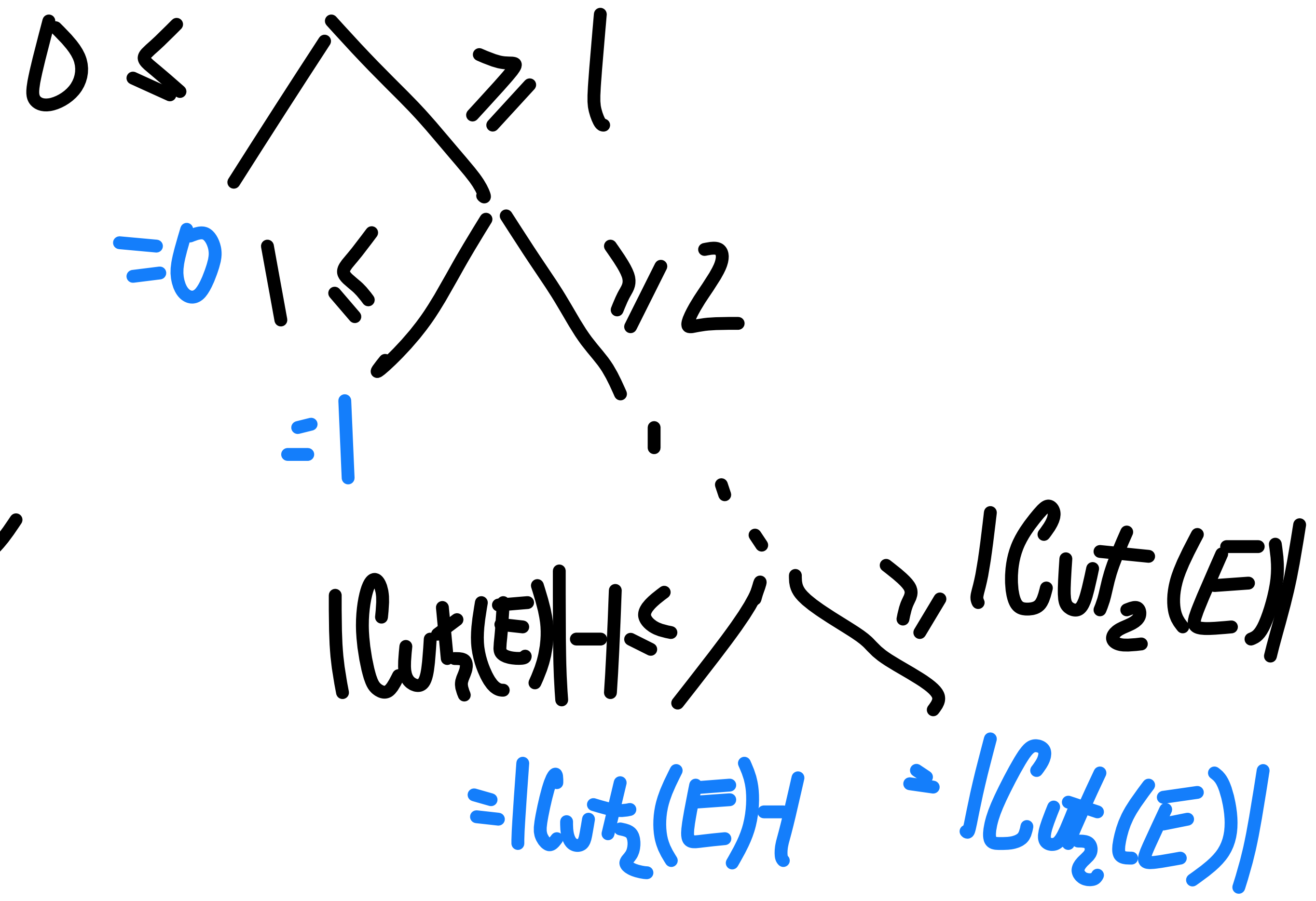

- Once parity of $Cut_2(E_1)$ and of $Cut_2(E_2)$ determined, recurse on unsat side

Refuting Systems of Equations in SP

Complexity: $O(\log n)$ recursive rounds.

At most $|Cut_2(E_1)| \cdot |Cut_2(E_2)| \leq n^2$ queries per round

$\therefore$ $n^{O(\log n)}$ size

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Thm: Tseitin requires $\Omega(n)$ depth to refute in Semantic CP

Semantic CP

Semantic CP refutation of $P = \{Ax \geq b\} \subseteq [0,1]$:

$a_1 x \geq b_1$ $\quad$ $a_2 x \geq b_2$ $\quad \ldots \quad$ $a_m x \geq b_m$

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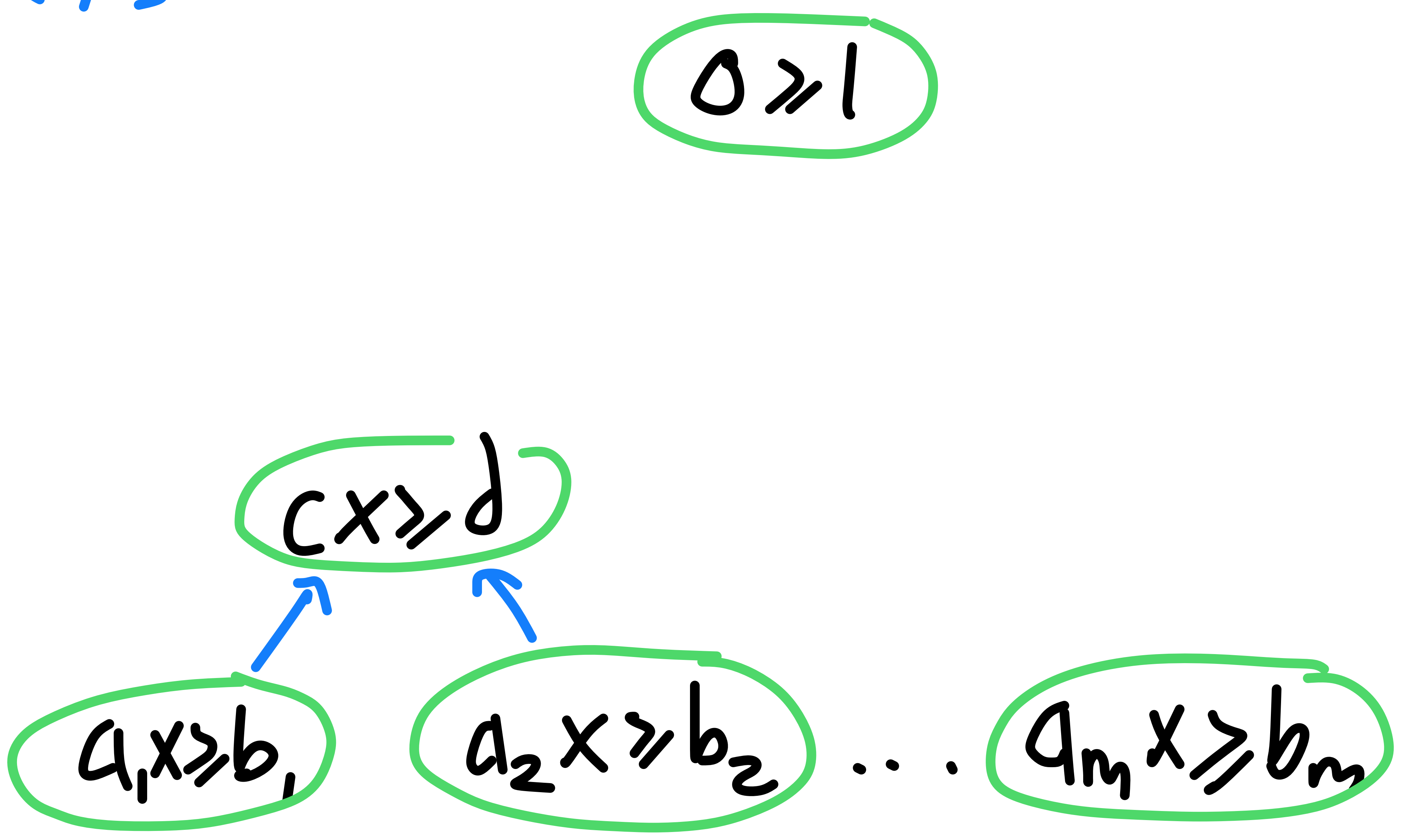

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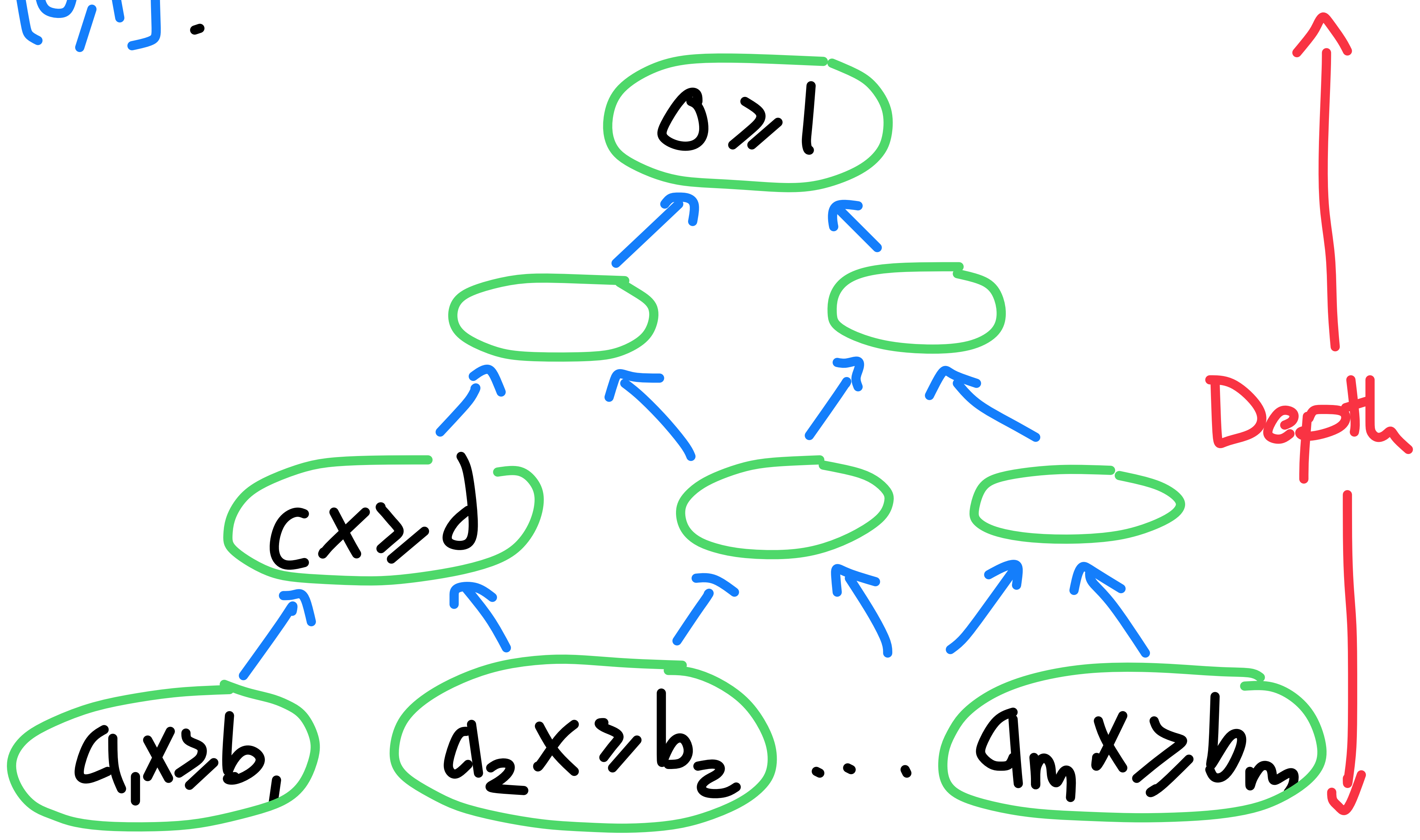

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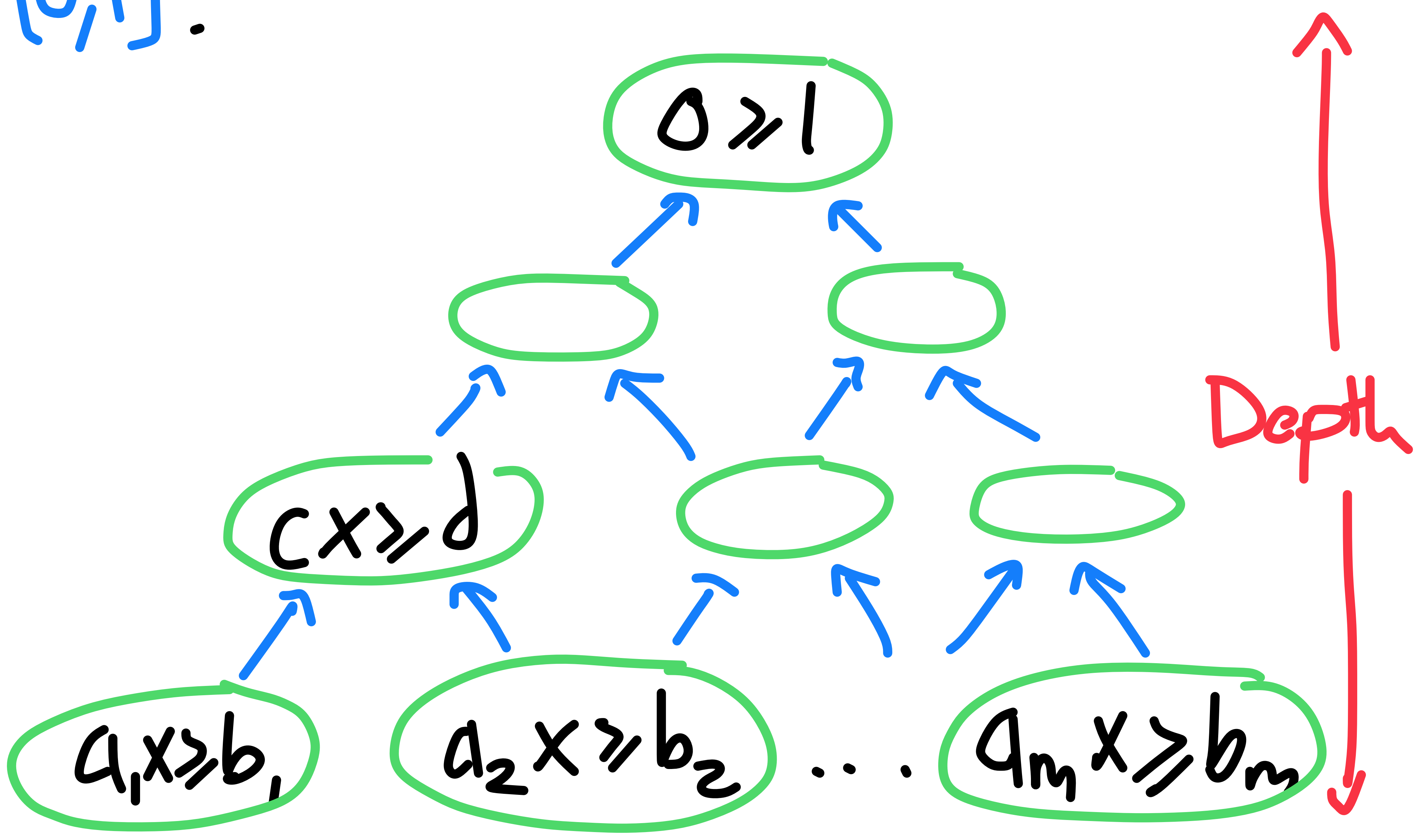

Similar for CP, semantic deduction replaced by CG cut

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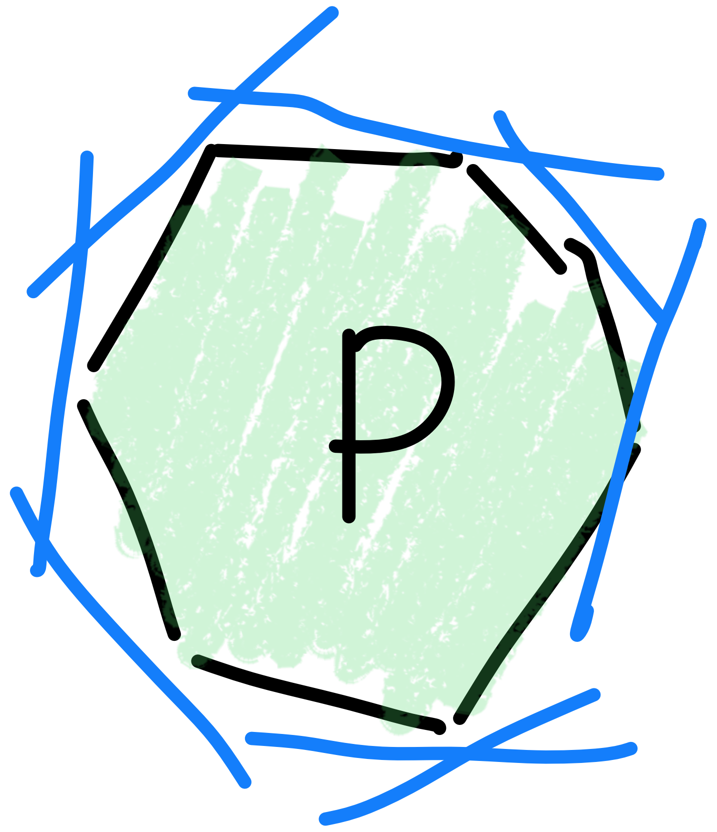

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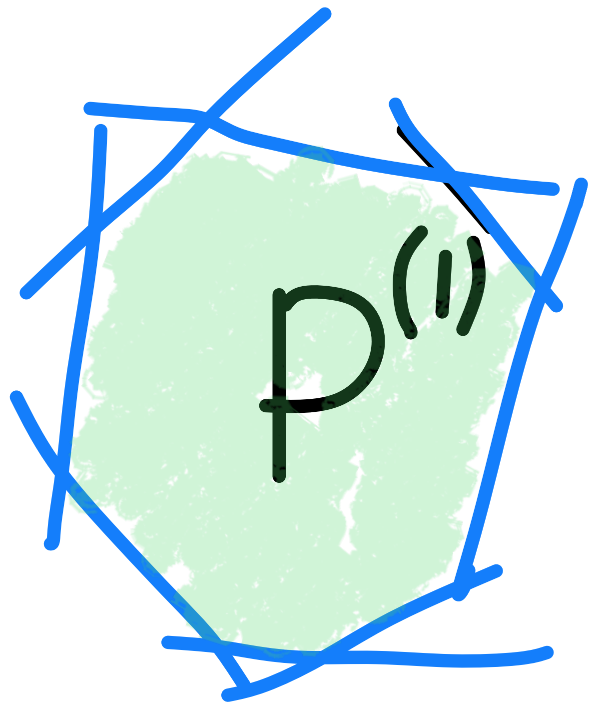

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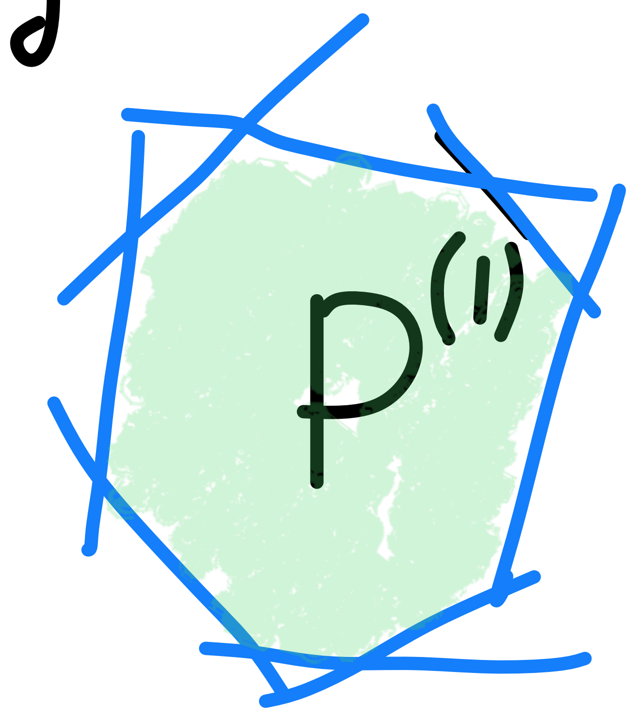

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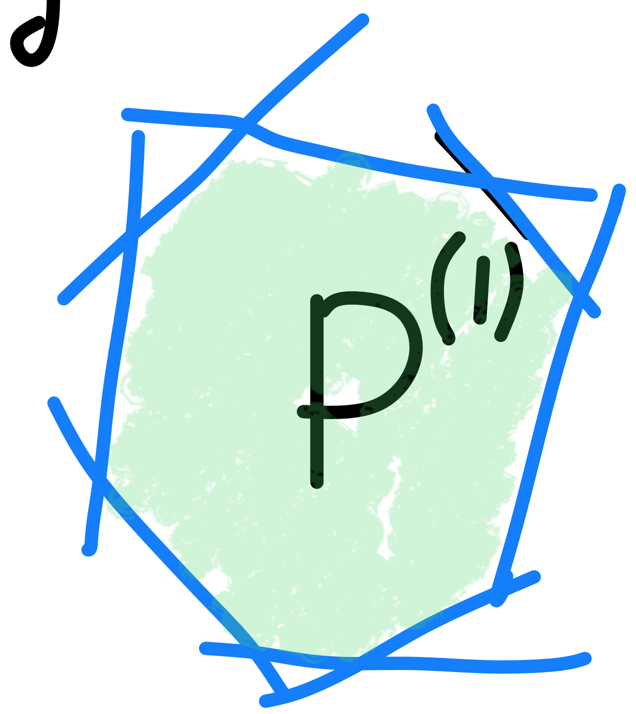

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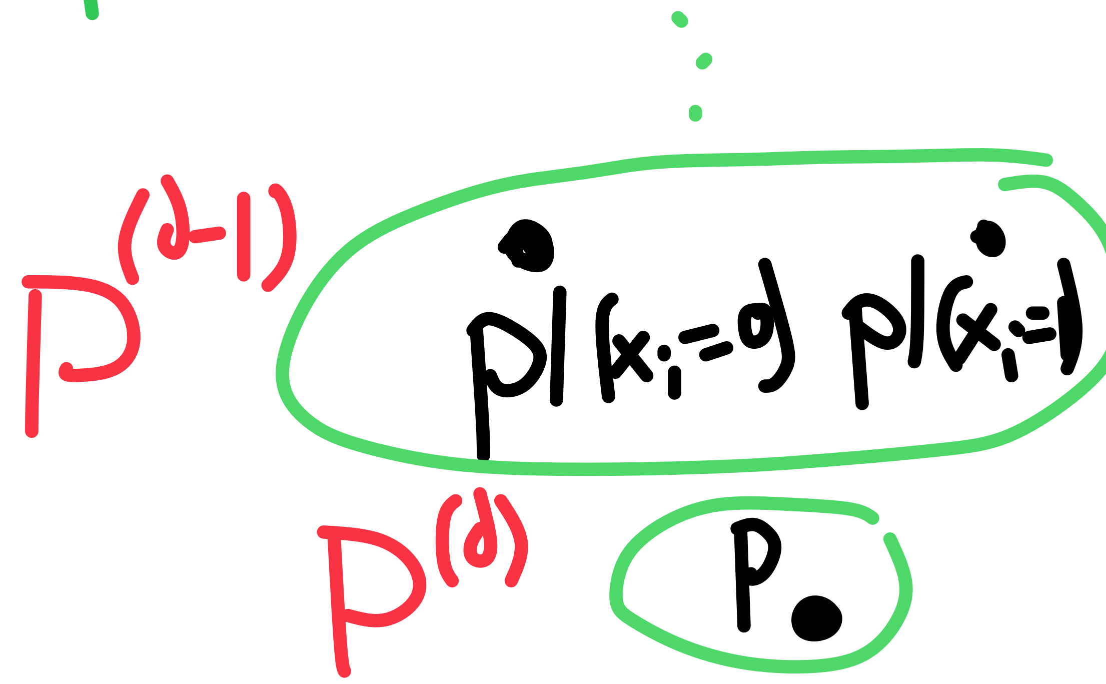

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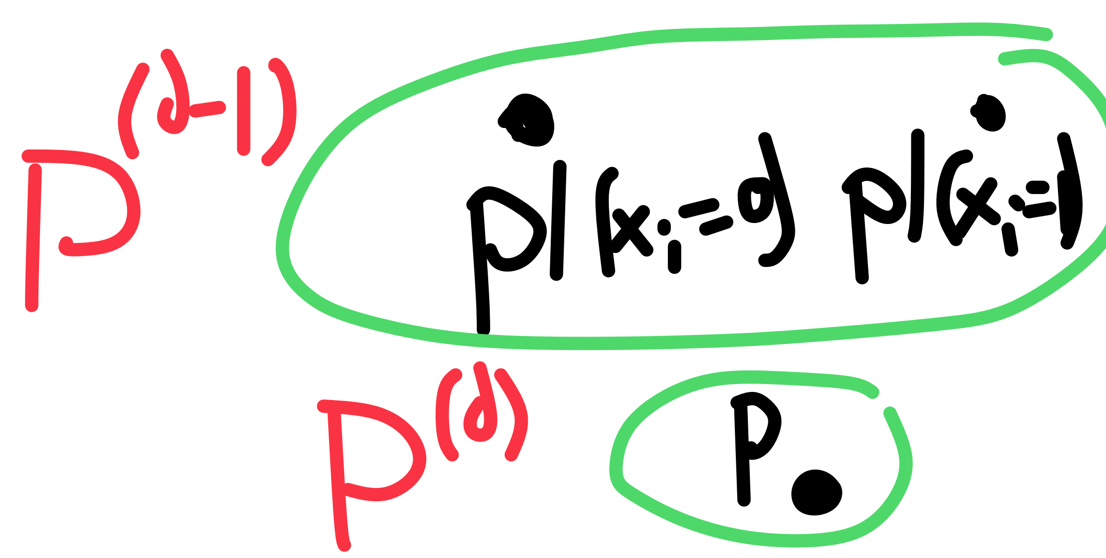

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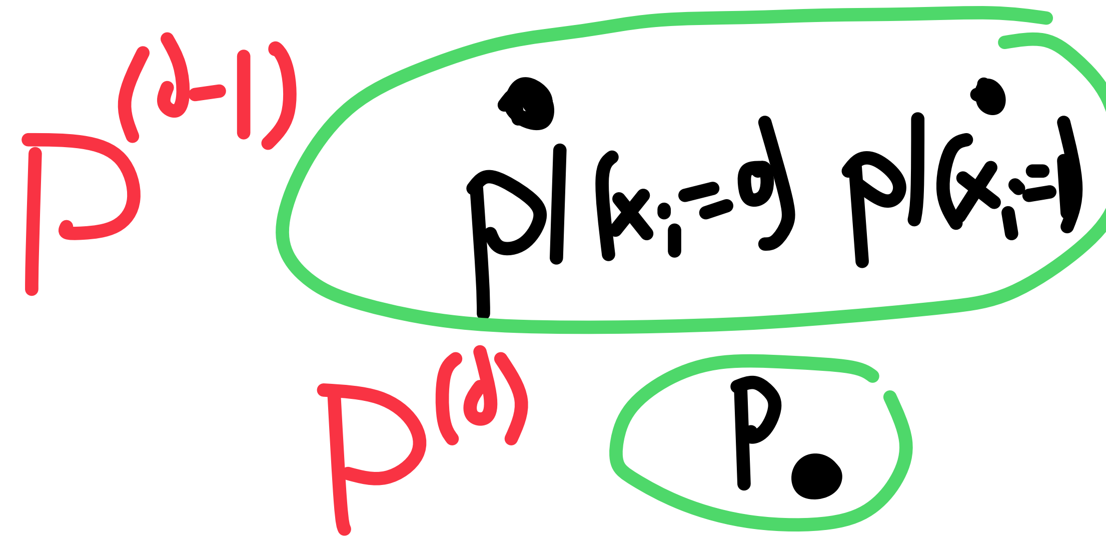

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block 1 block 2 block n

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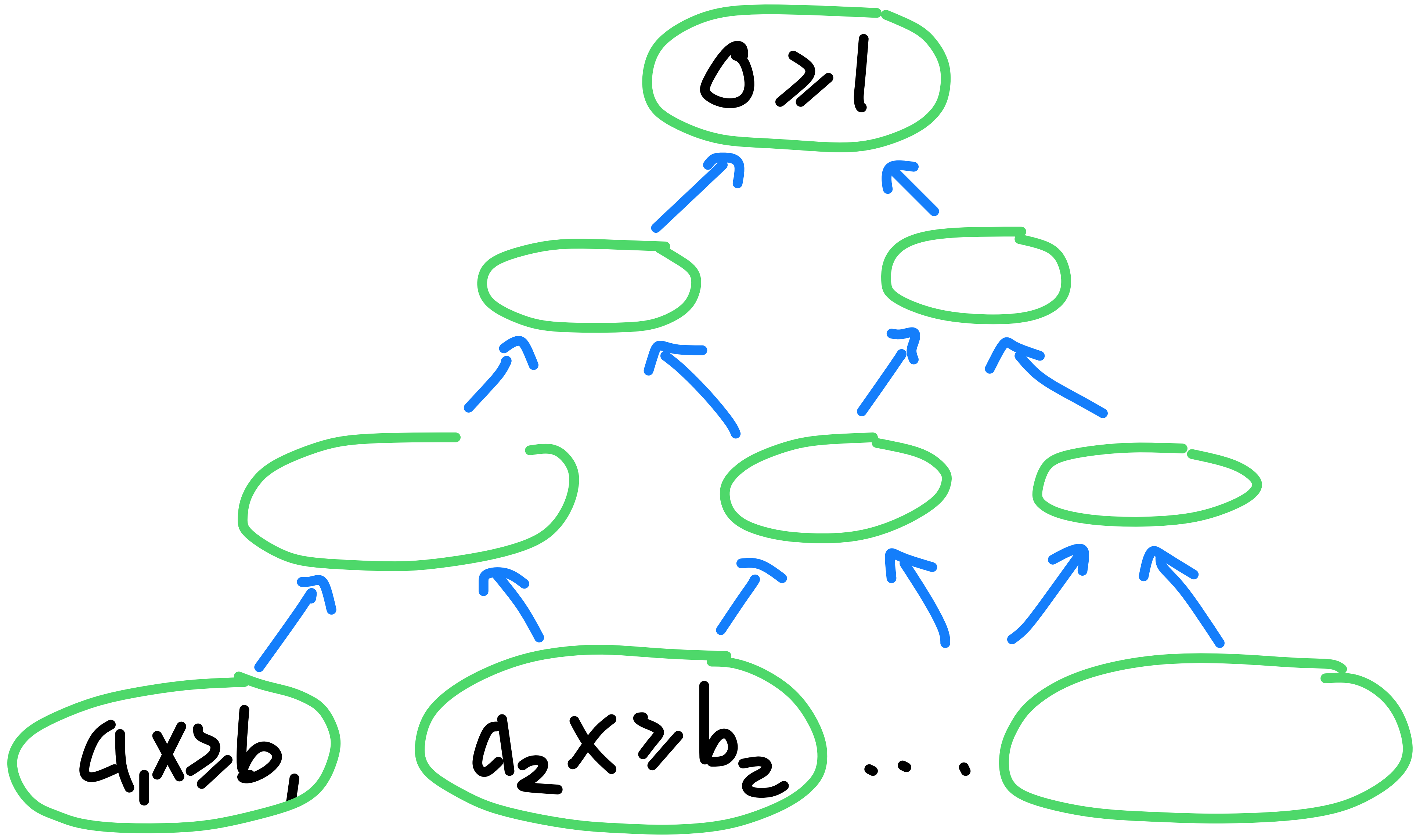
→ lower bound on Tseitin is proved directly by replacing lifting with expansion

Depth Bounds for Semantic CP

Goal: Find a long root-to-leaf path in the proof



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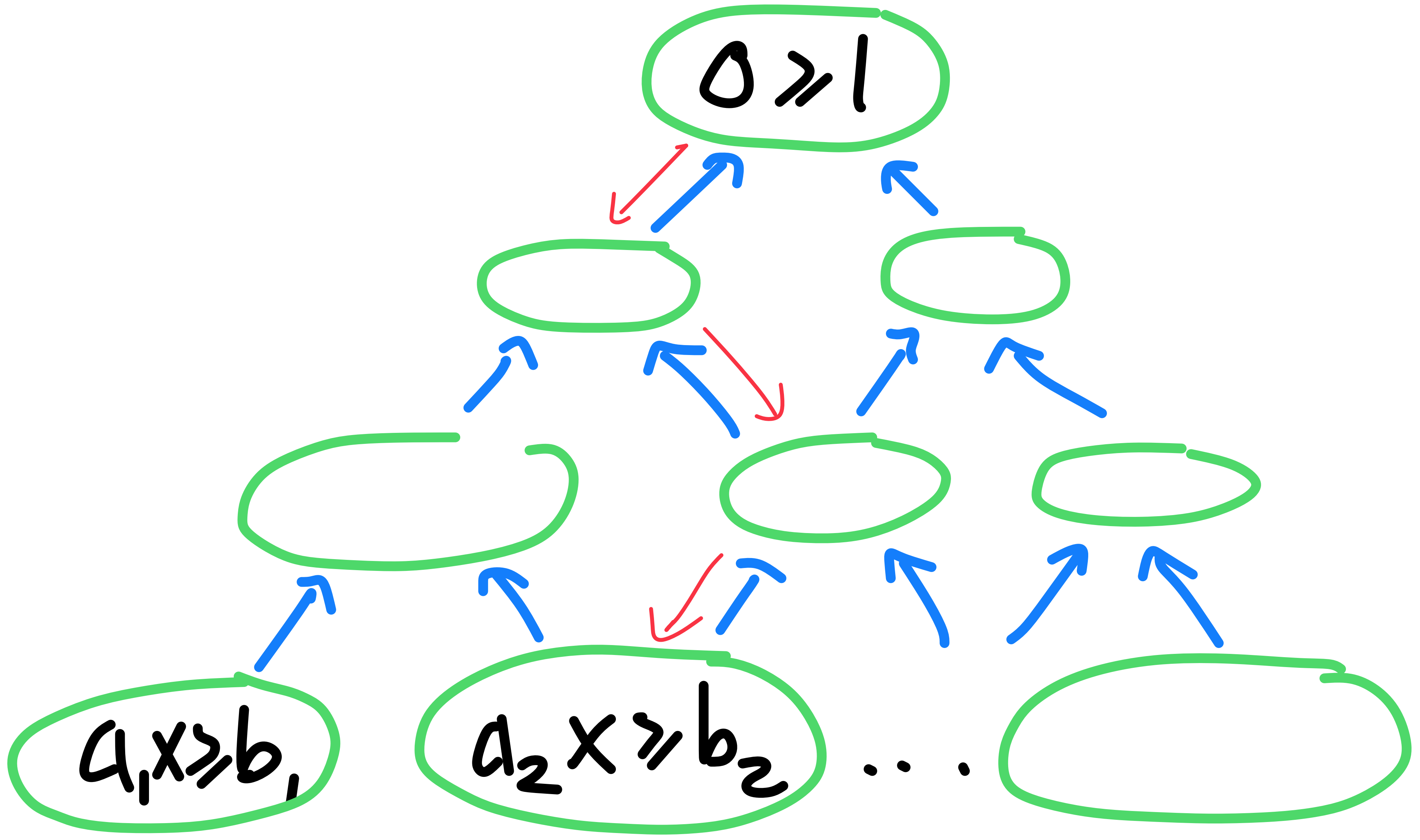

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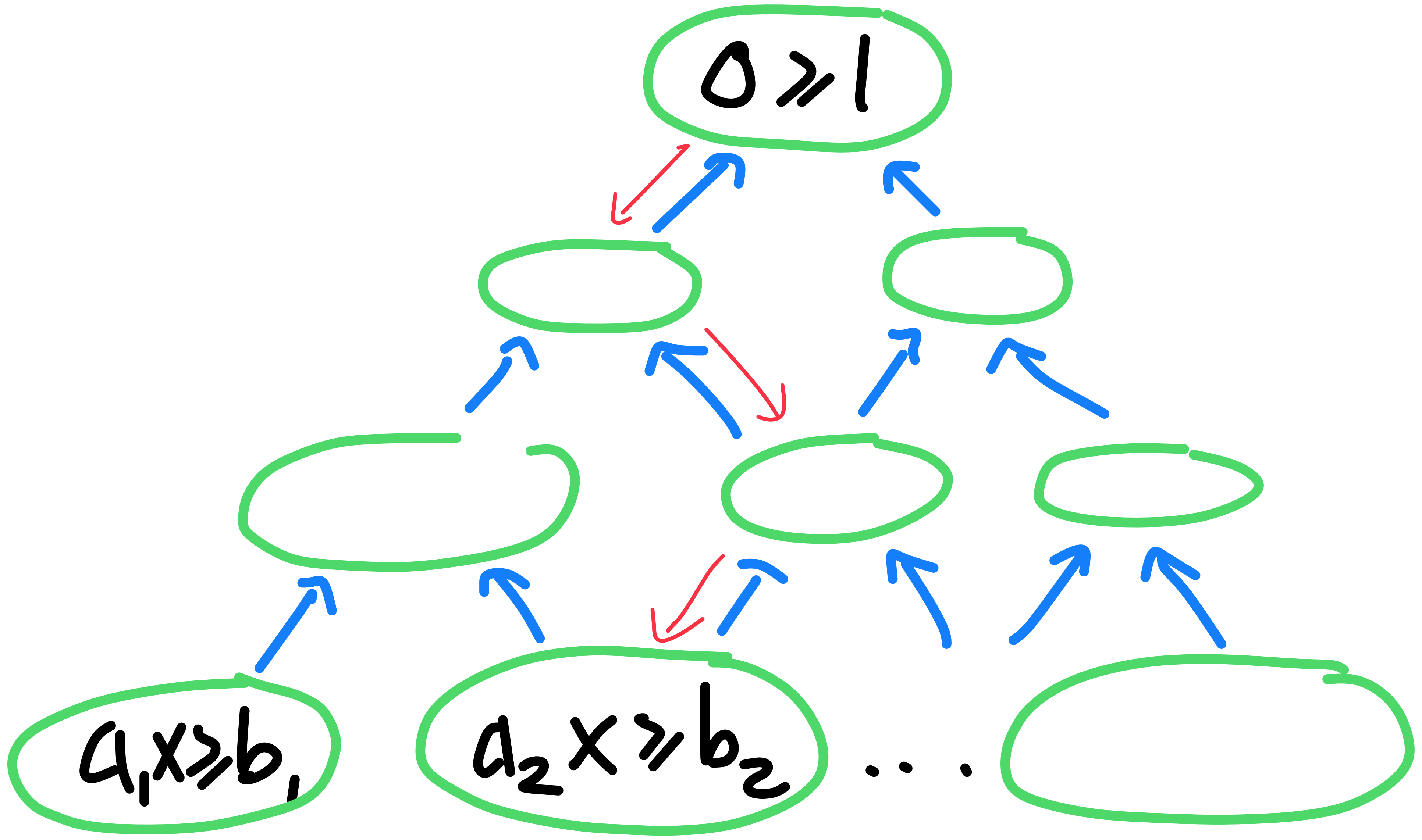

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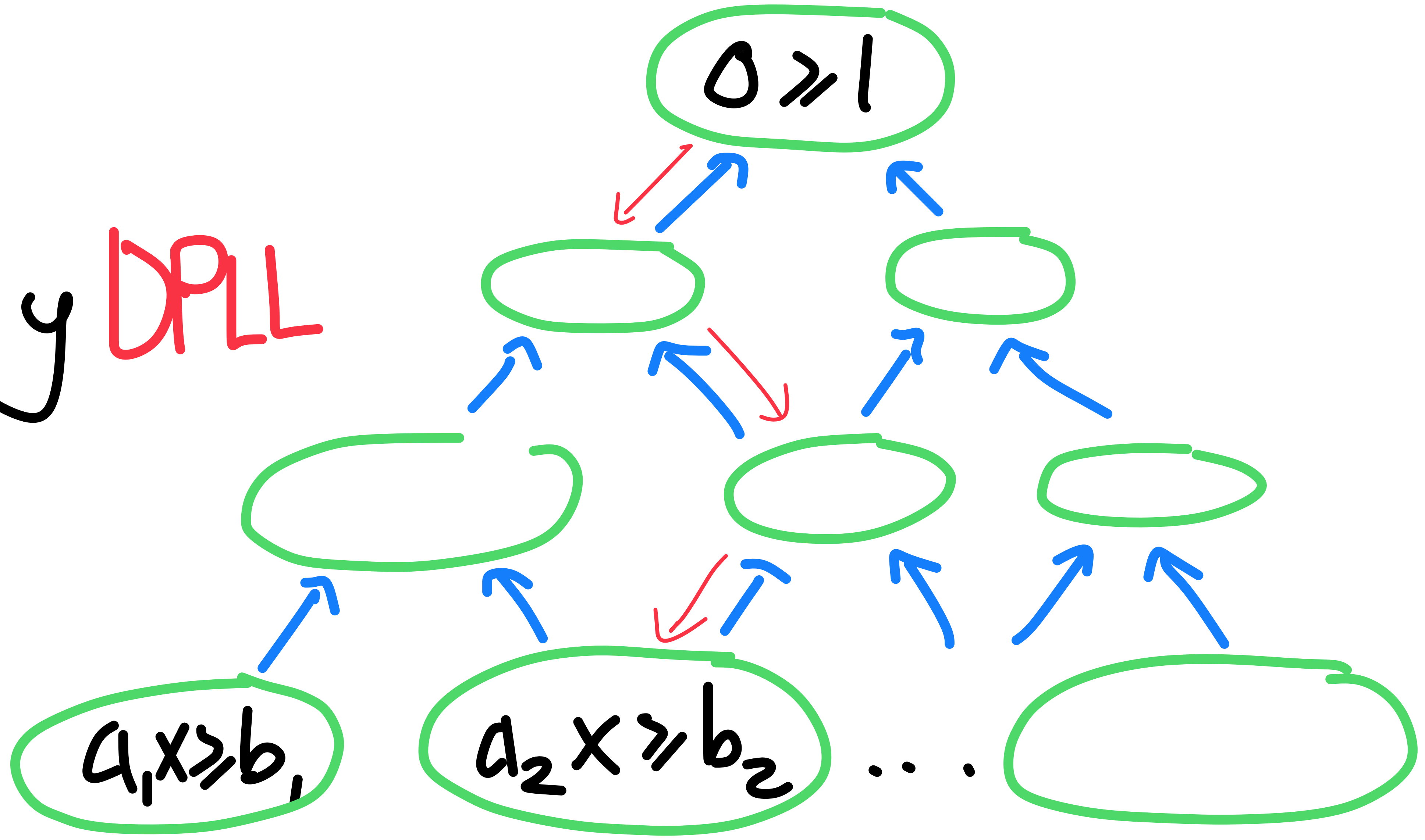

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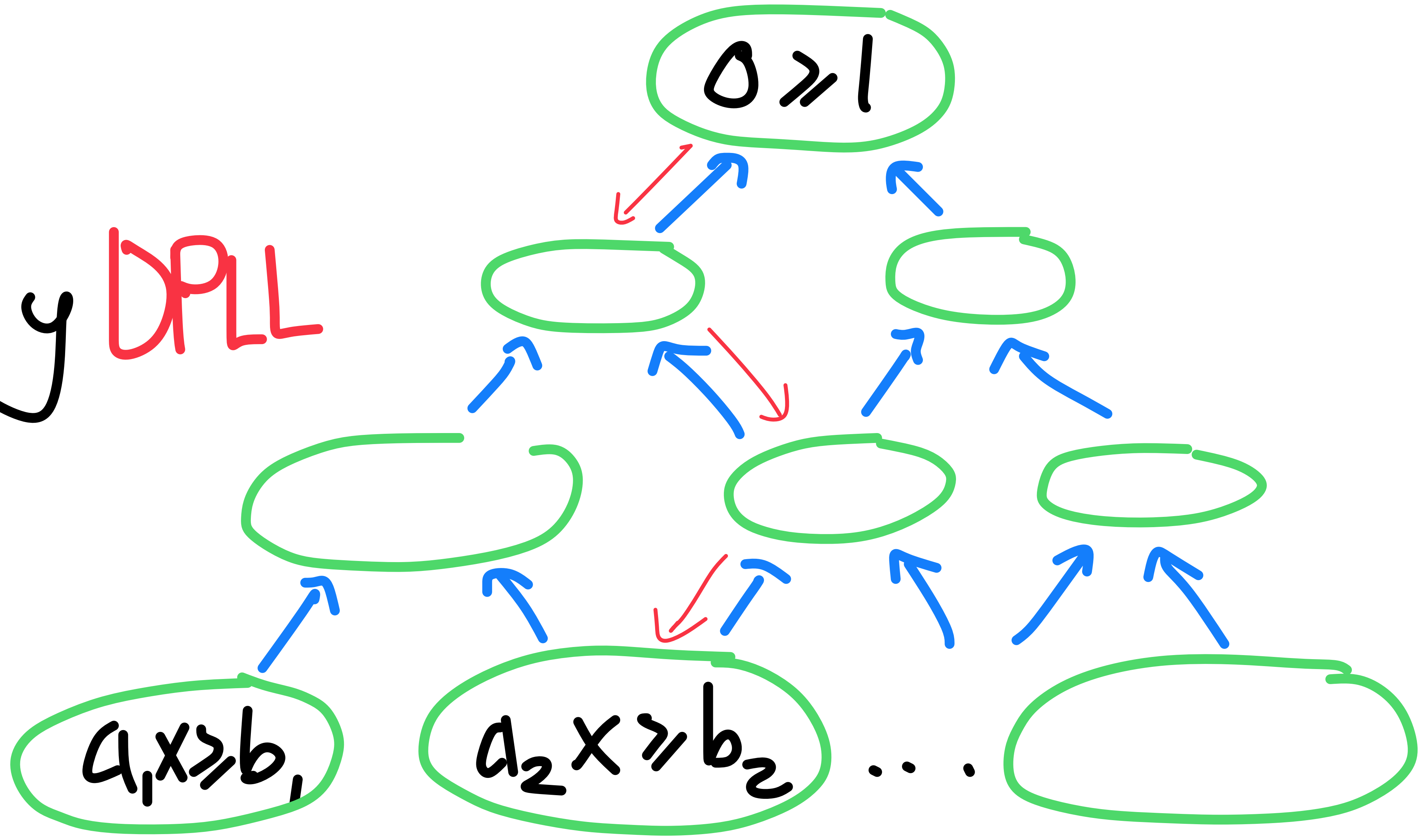

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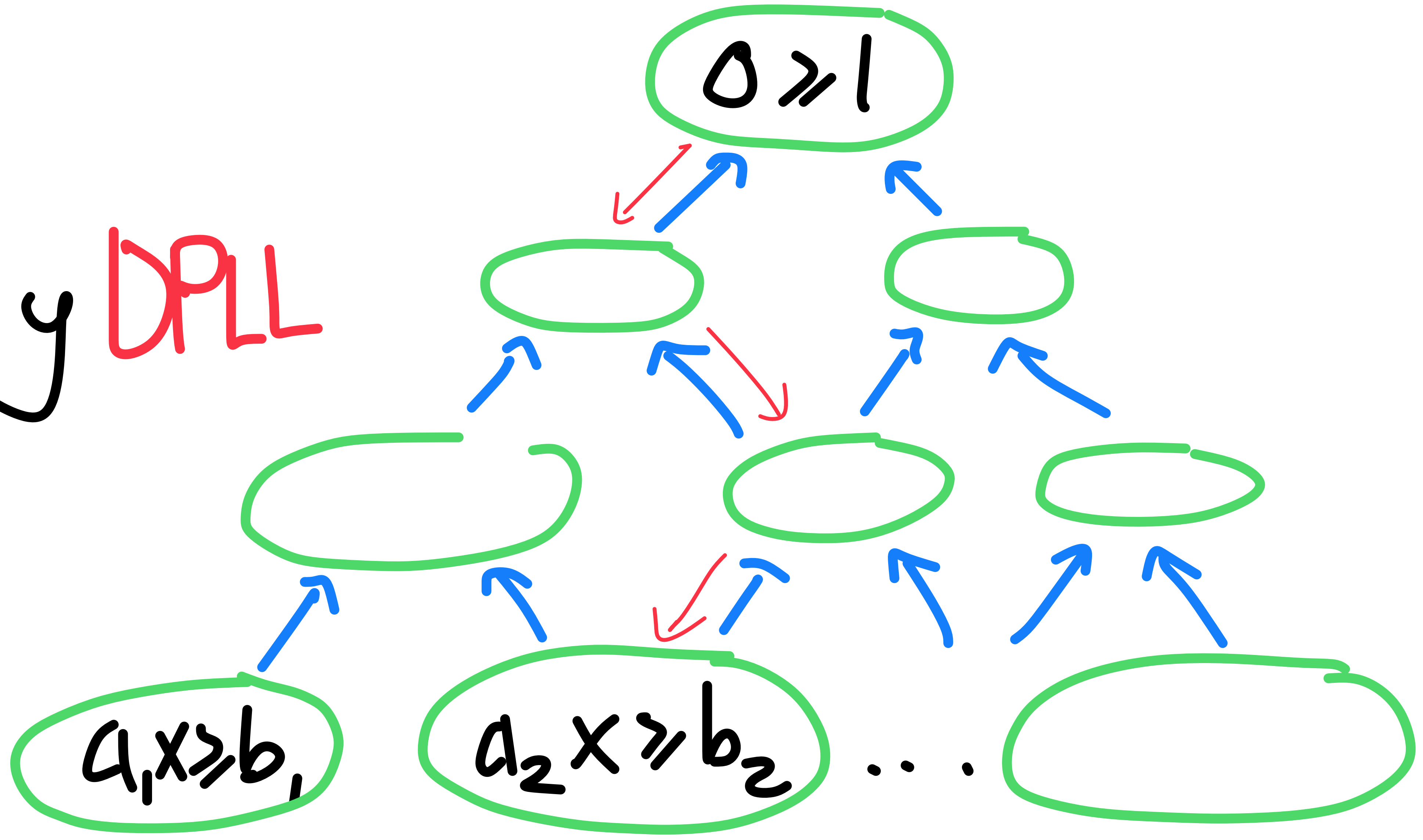

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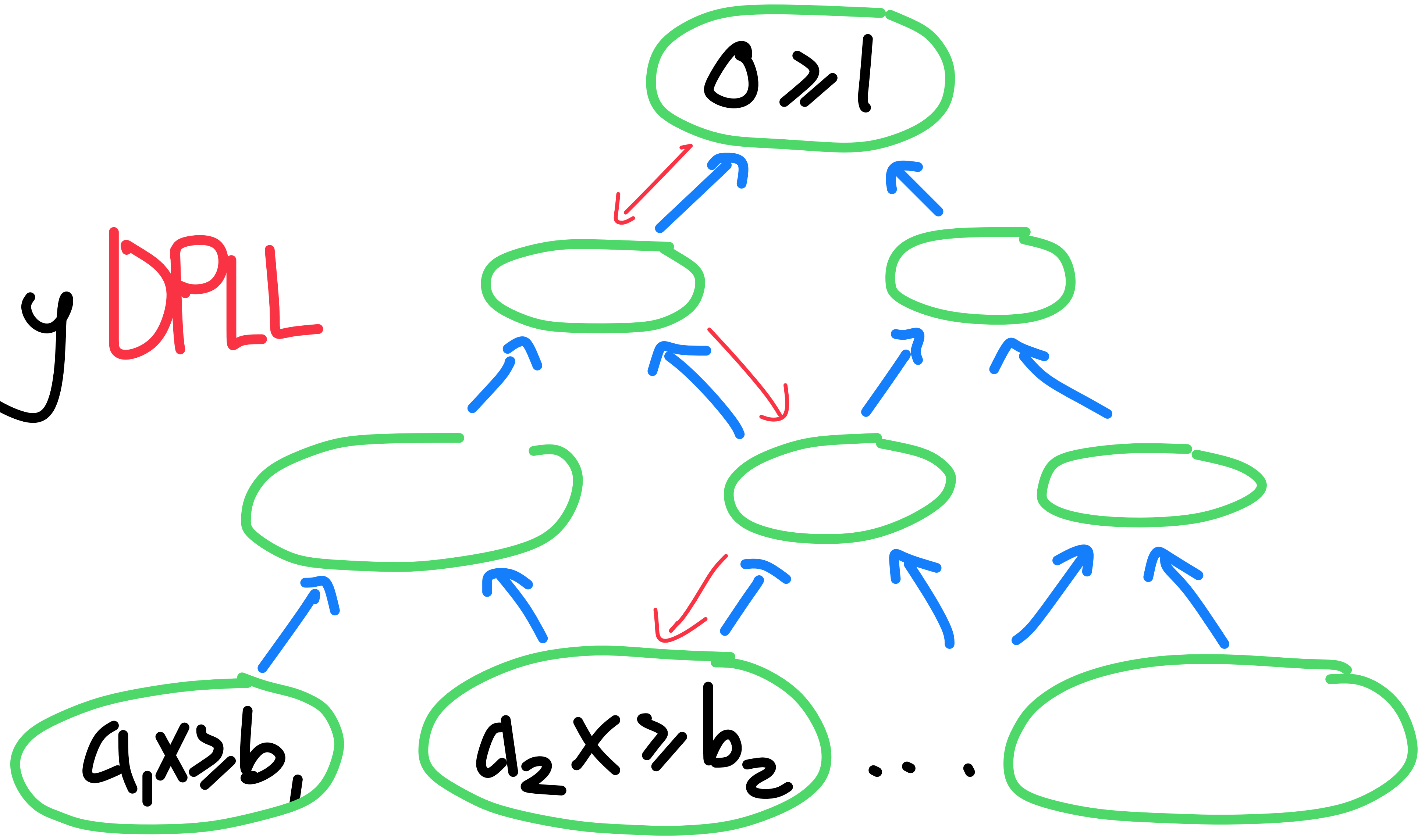

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$H|\rho$ is good then

- H is not a leaf
- We can find a restriction ρ' such that one of the children H_1, H_2 of H is good under ρ'

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Technical Lemma: Let H_1, H_2 be the children of H and $H \restriction \rho$ be good. Then we can obtain ρ' by fixing 2 additional bits to $\{0,1\}$ s.t. $H_1 \restriction \rho'$ or $H_2 \restriction \rho'$ is good

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Maintain the following invariants for H (initially $0 \geq 1$) and ρ (initially $*^n$)

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Maintain the following invariants for H (initially $0 \geq 1$) and ρ (initially $*^n$)
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Say halfspace H is good if $H(\frac{1}{2}^n) = 0$

Technical Lemma: Let H_1, H_2 be the children of H and $H \upharpoonright \rho$ be good. Then we can obtain ρ' by fixing 2 additional bits to $\{0,1\}$ s.t. $H_1 \upharpoonright \rho'$ or $H_2 \upharpoonright \rho'$ is good

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Claim: Let $H \subseteq \mathbb{R}^n$ be good. For any $I \subseteq [n]$, $b \in \{0,1\}$, there is an assignment π to the variables x_i, $i \in I$ such that

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block 1, block 2, block n

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block 1 block 2 block n

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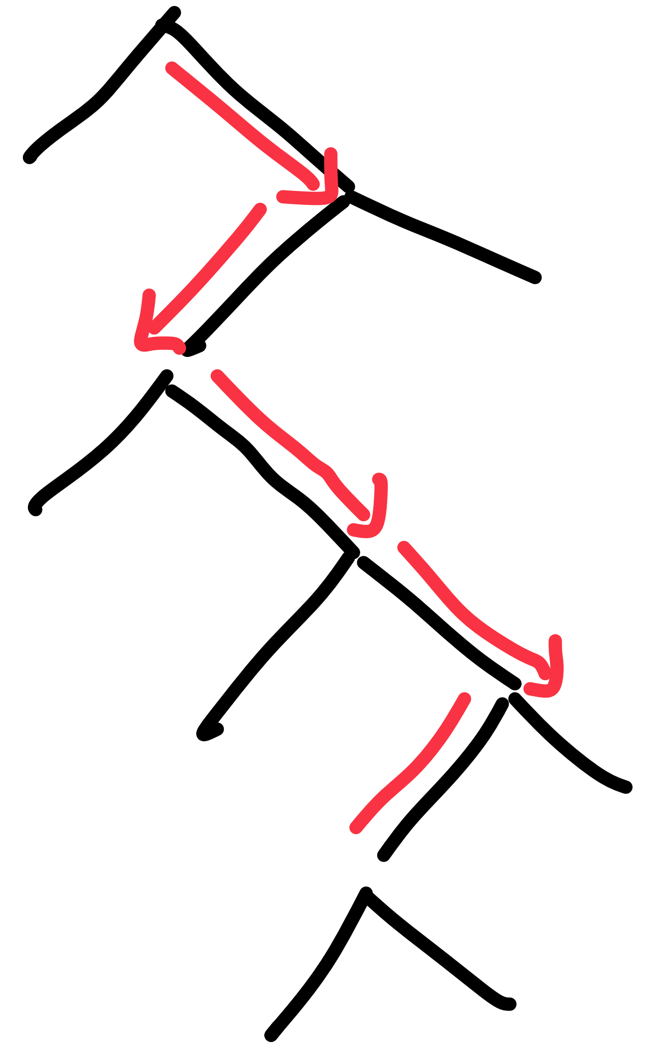

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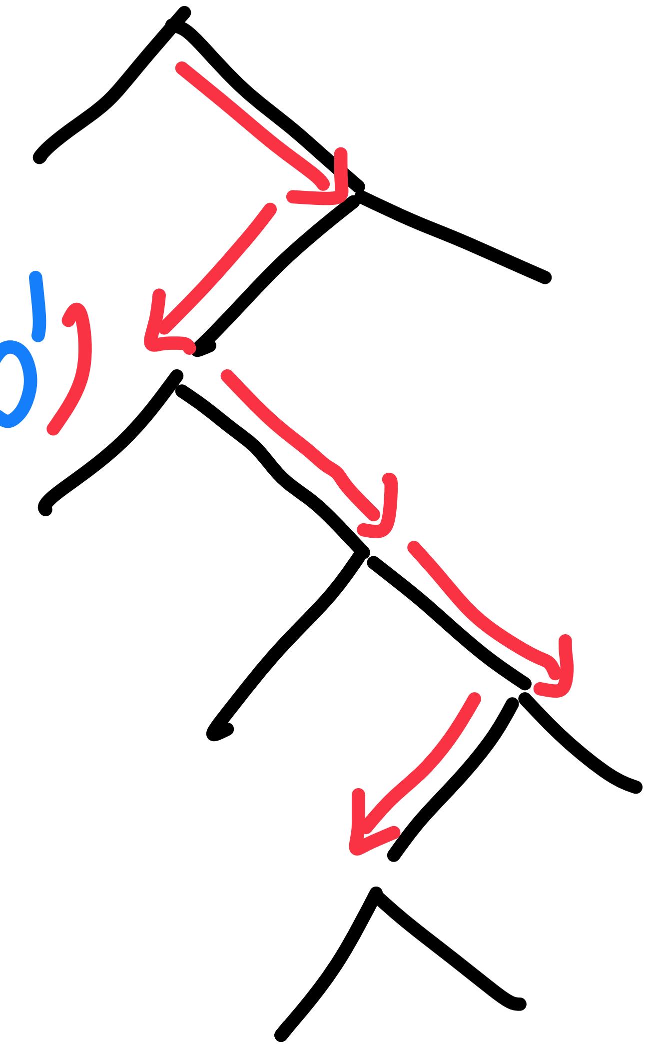

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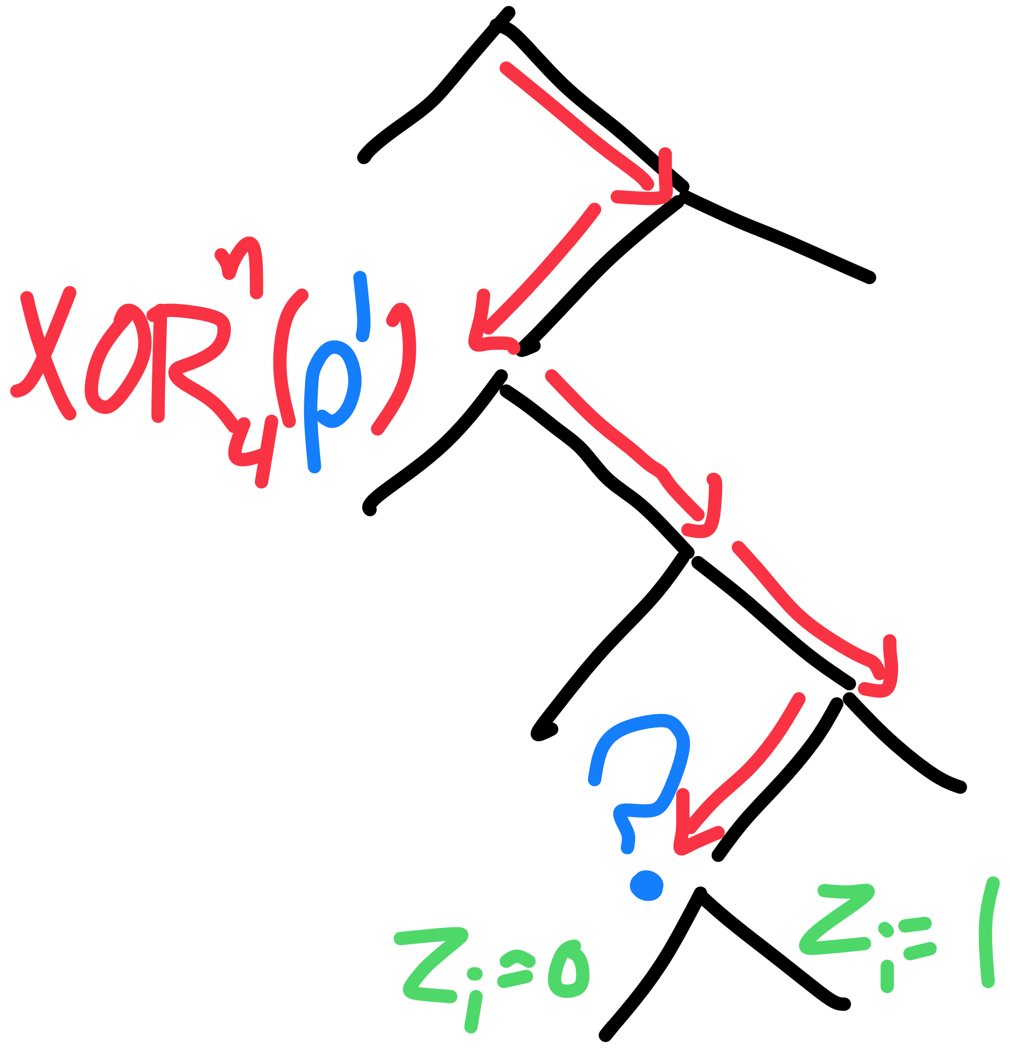

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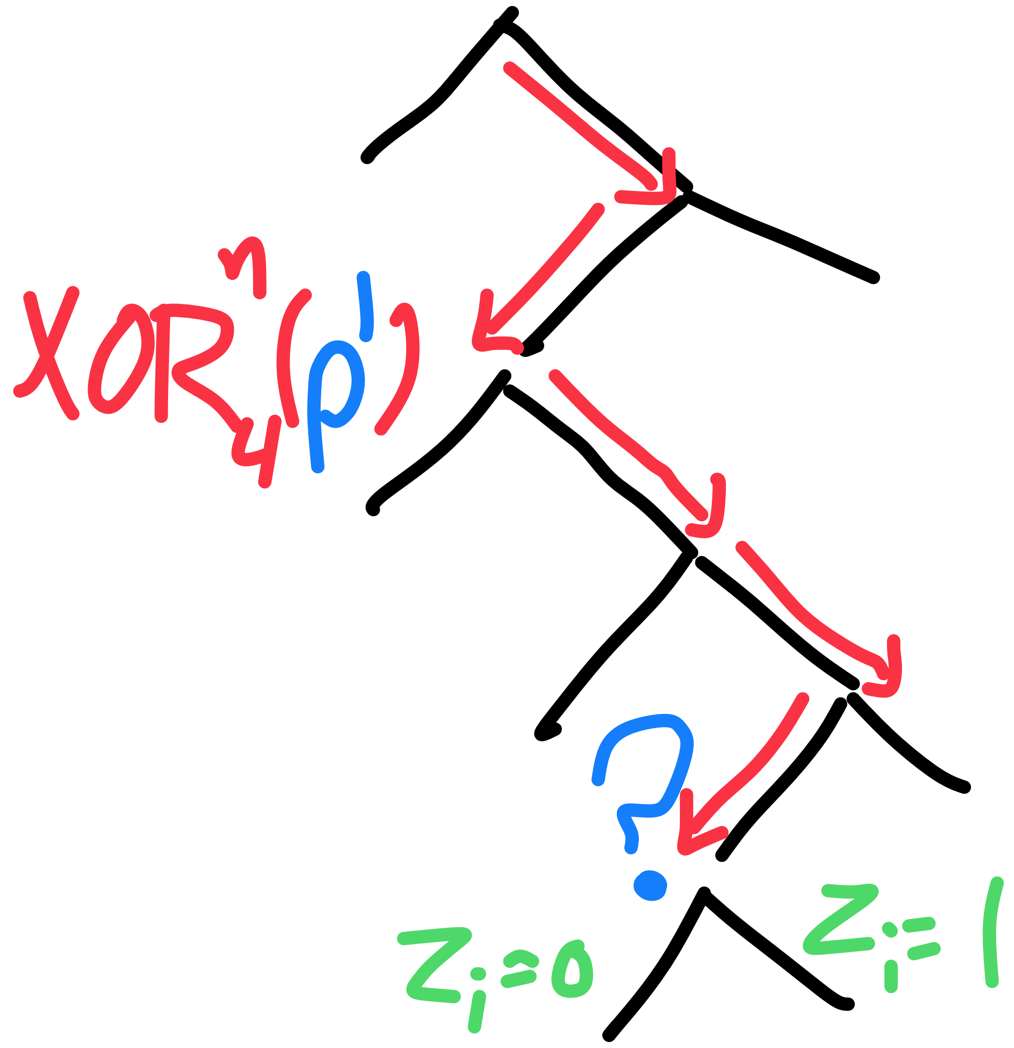

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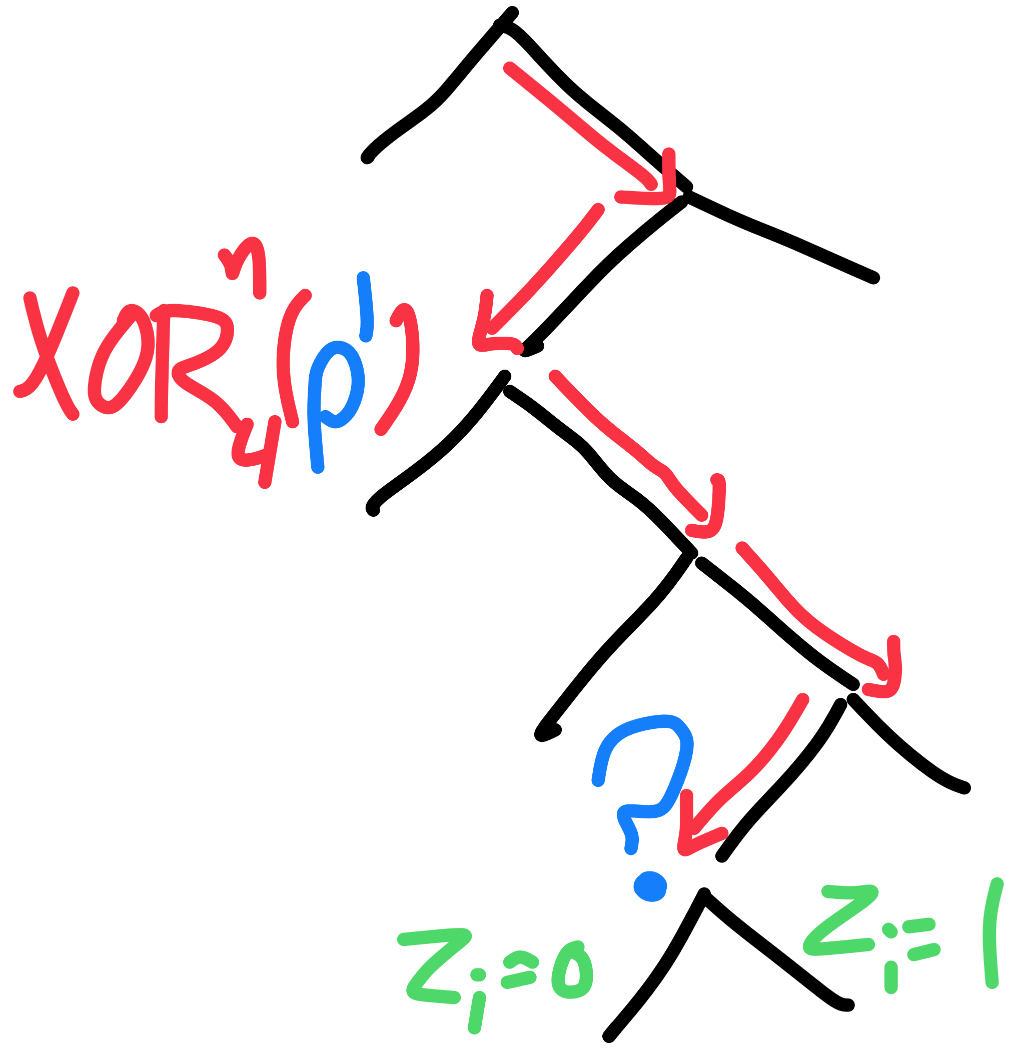

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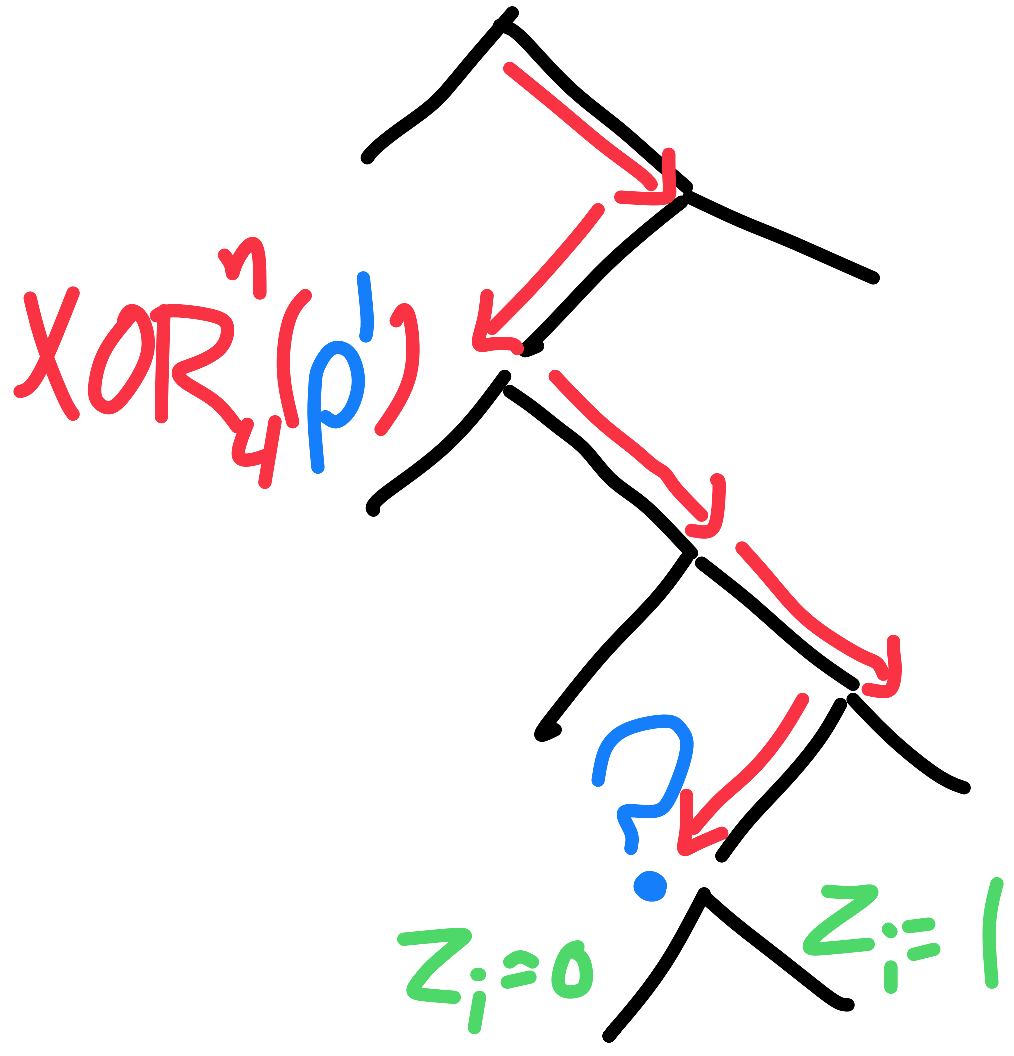

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$$\therefore D_{sCP}(F \circ XOR_4^n) \geq D_{DPLL}(F)/2$$

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Claim: Let $H \subseteq \mathbb{R}^n$ be good. For any $I \subseteq [n]$, $b \in \{0,1\}$, there is an assignment π to the variables x_i, $i \in I$ such that

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e.g. $b=0$, $3x_1 - 2x_2 - x_3 + x_4 \geq 2$

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"Setting $a_1, \ldots, a_{k-1}$ to compensate for a_k"

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e.g. $b=0$, $I = \{1,2,3\}$

$$\begin{array}{cccc} 3x_1 & -2x_2 & -x_3 & +x_4 \geq 2 \\ \uparrow & \uparrow & \uparrow & \\ 0 & \underline{1} & \underline{1} & \end{array}$$

Wts $H \restriction \pi$ is good

"Setting $a_1, \ldots, a_{k-1}$ to compensate for a_k"

Depth Bounds for Semantic CP

Claim: Let $H \subseteq \mathbb{R}^n$ be good. For any $I \subseteq [n]$, $b \in \{0,1\}$, there is an assignment π to the variables x_i, $i \in I$ such that

- $\bigoplus_{i \in I} \pi(x_i) = b$
- $H \restriction \pi$ is good

Main tool: Monotonicity of halfspaces

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e.g. $b = 0$, $3x_1 - 2x_2 - x_3 + x_4 \geq 2$

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Technical Lemma: Let H_1, H_2 be the children of H and $H\lceil\rho$ be good. Then we can obtain ρ' by fixing 2 additional bits to $\{0,1\}$ s.t. $H_1\lceil\rho'$ or $H_2\lceil\rho'$ is good

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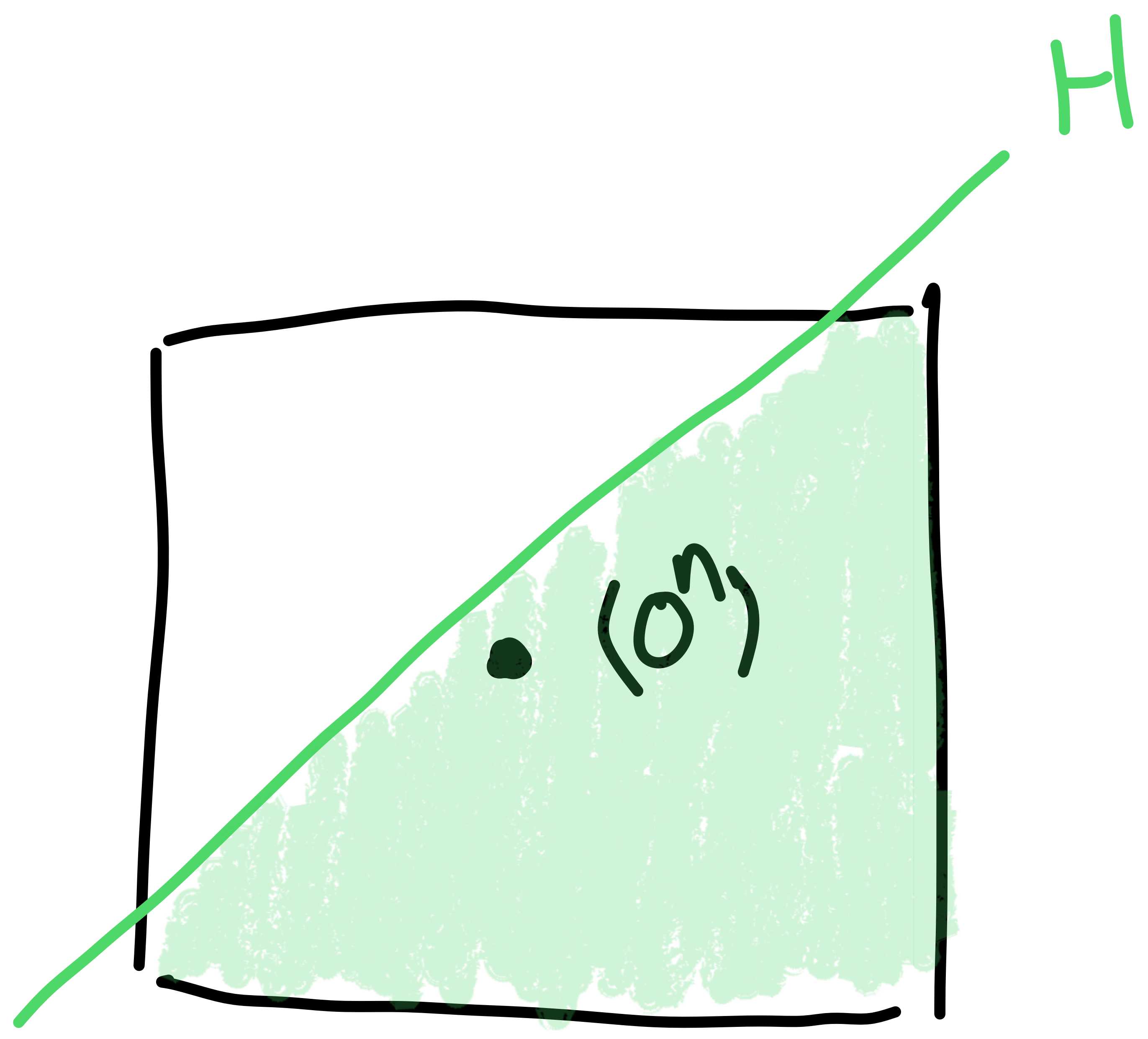
Suppose $\rho = *^n$



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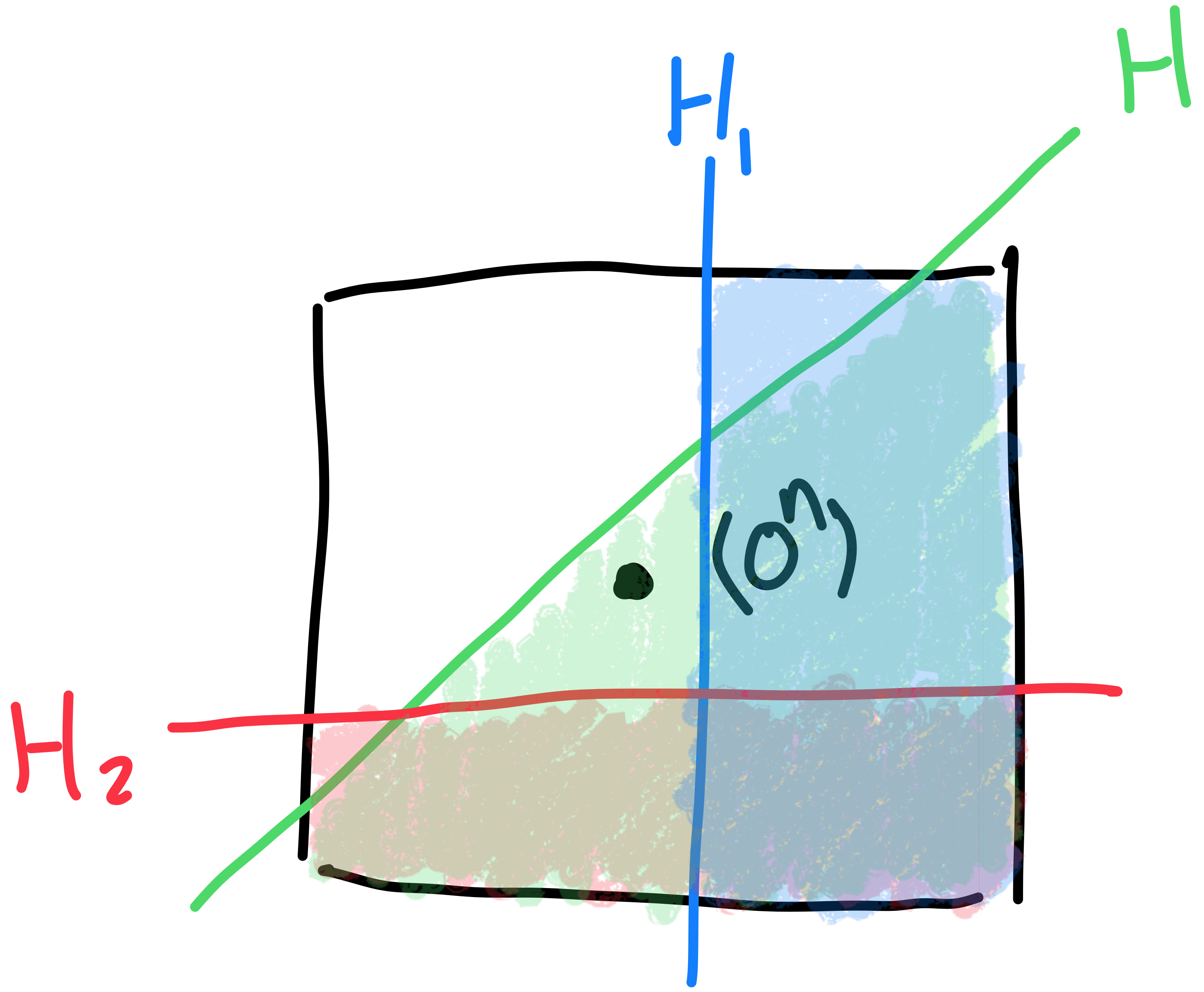

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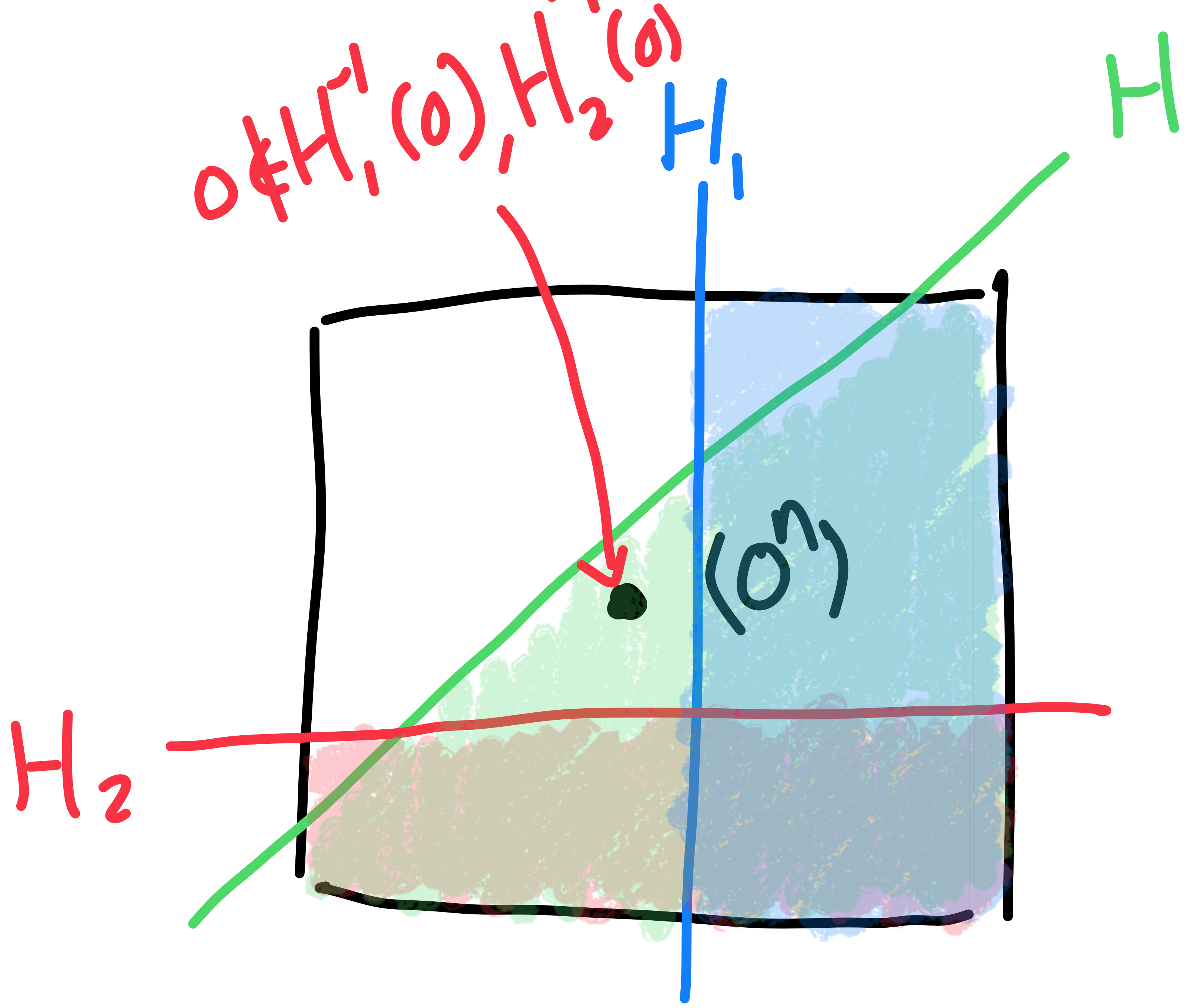

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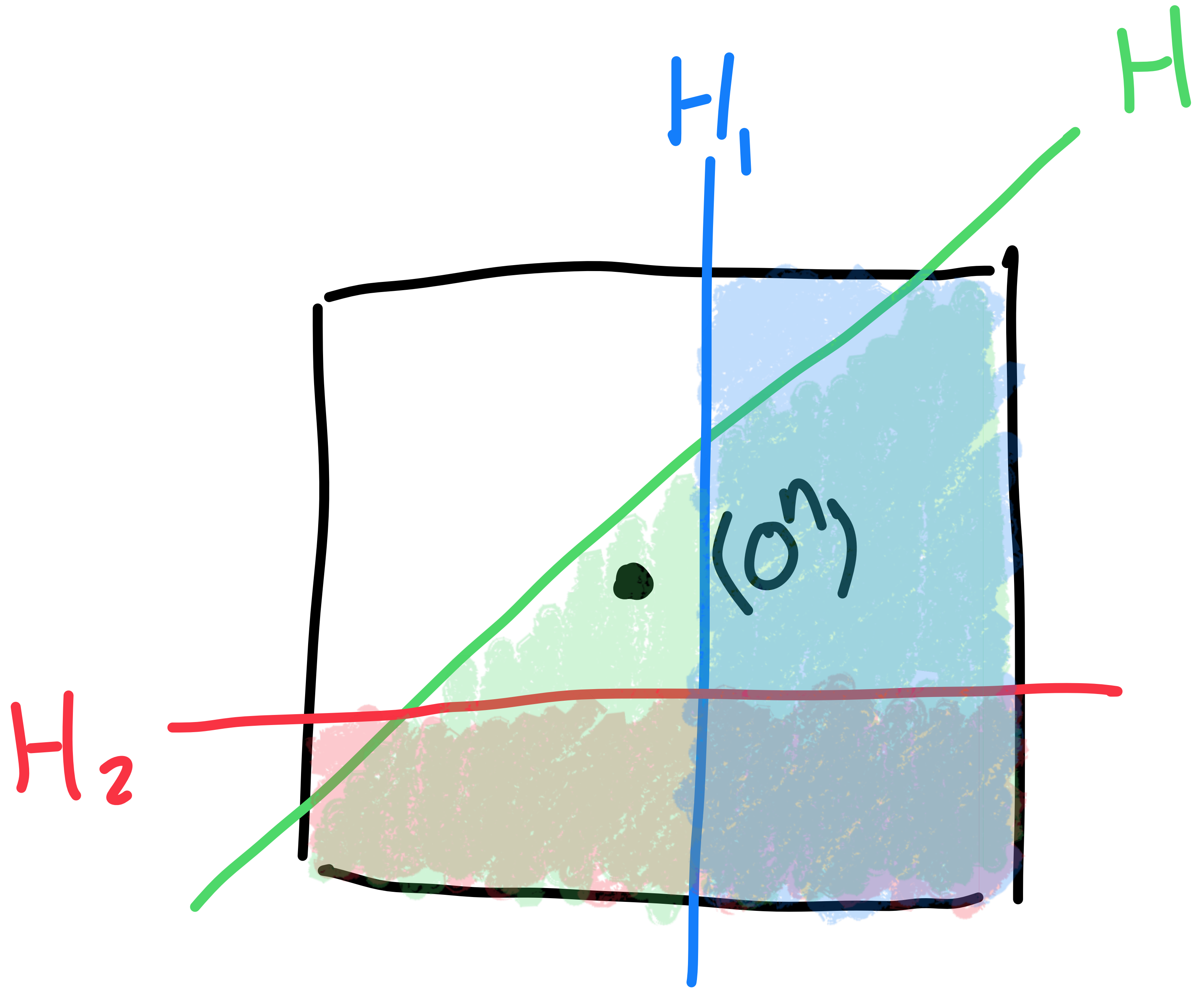

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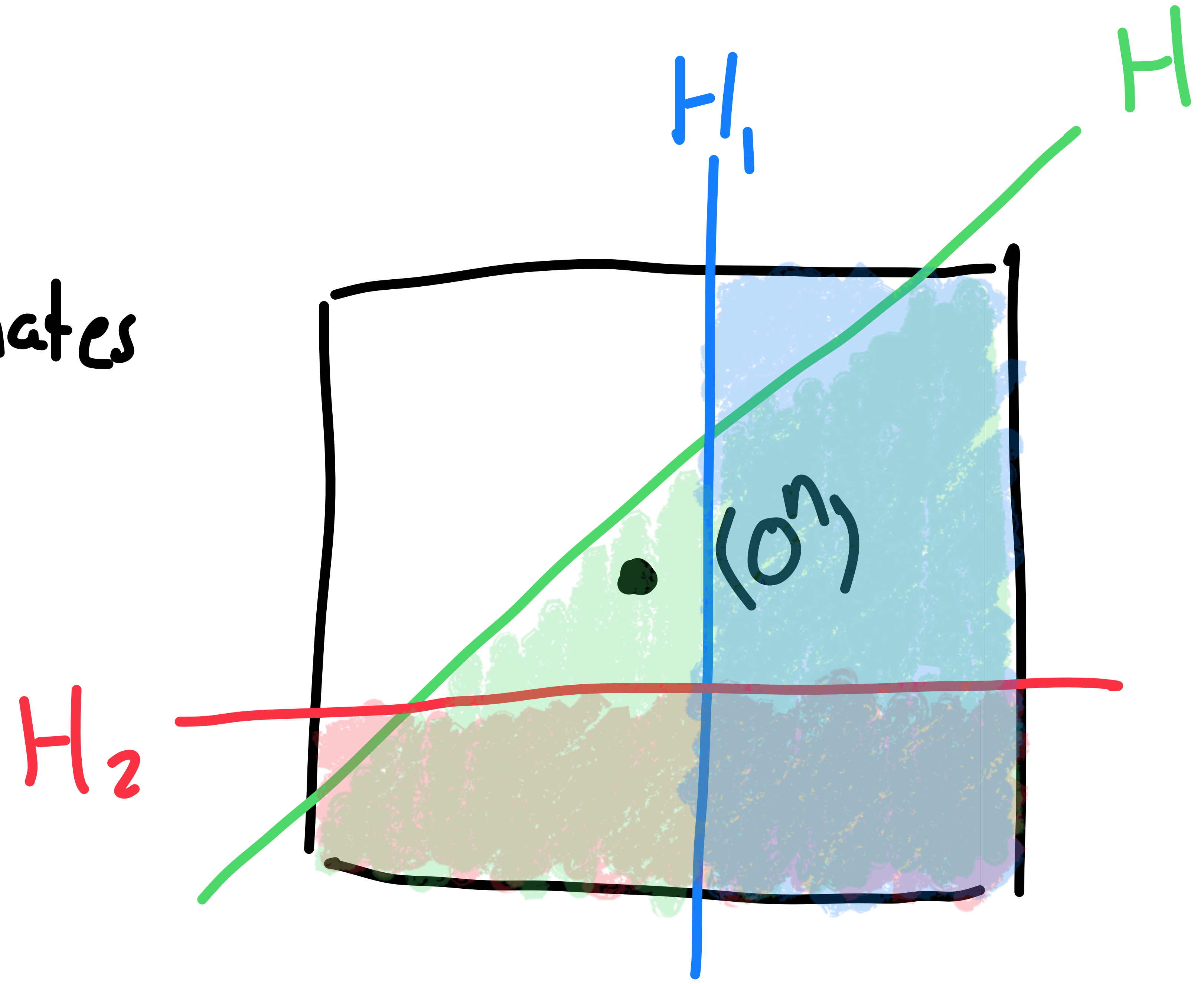

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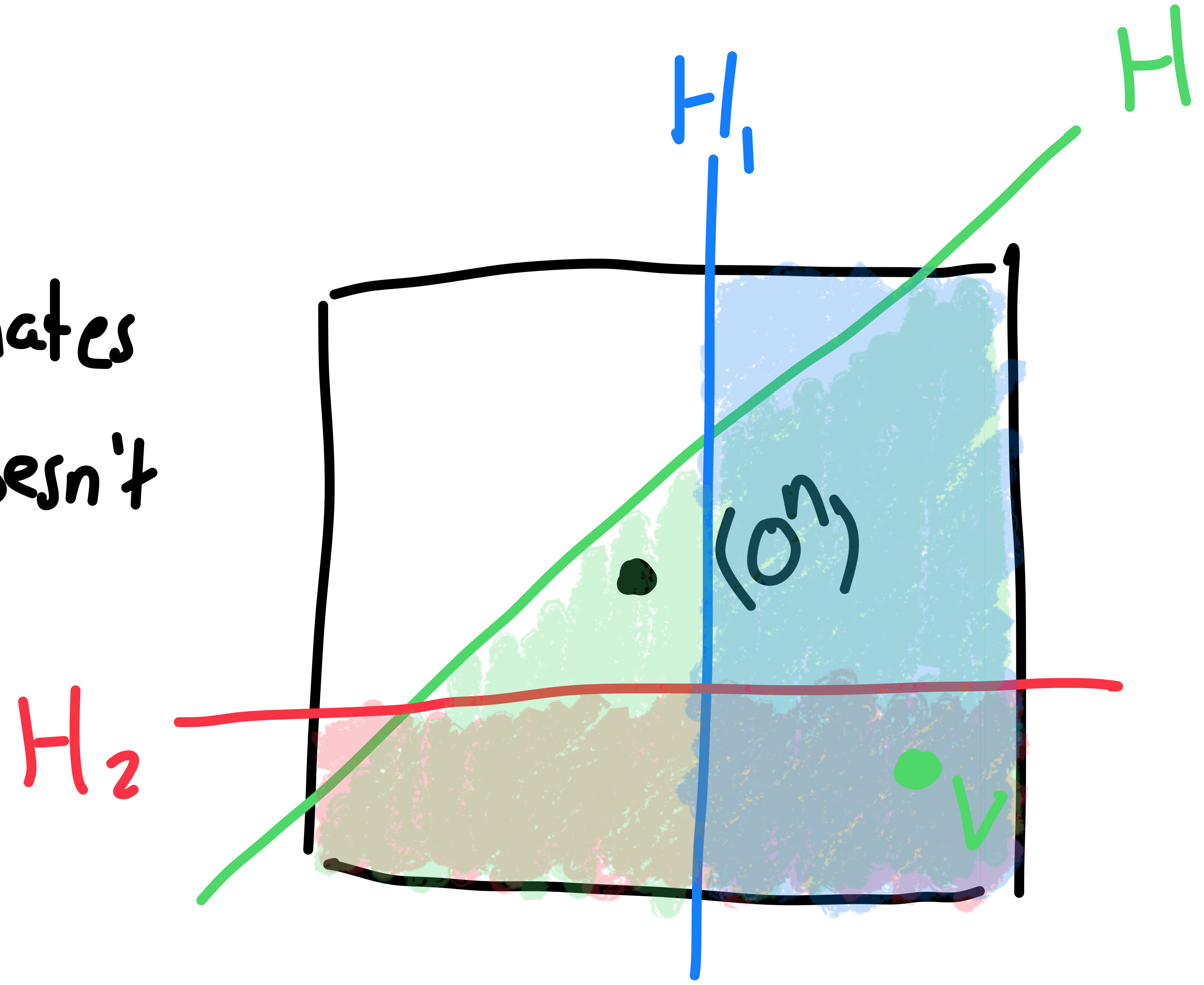

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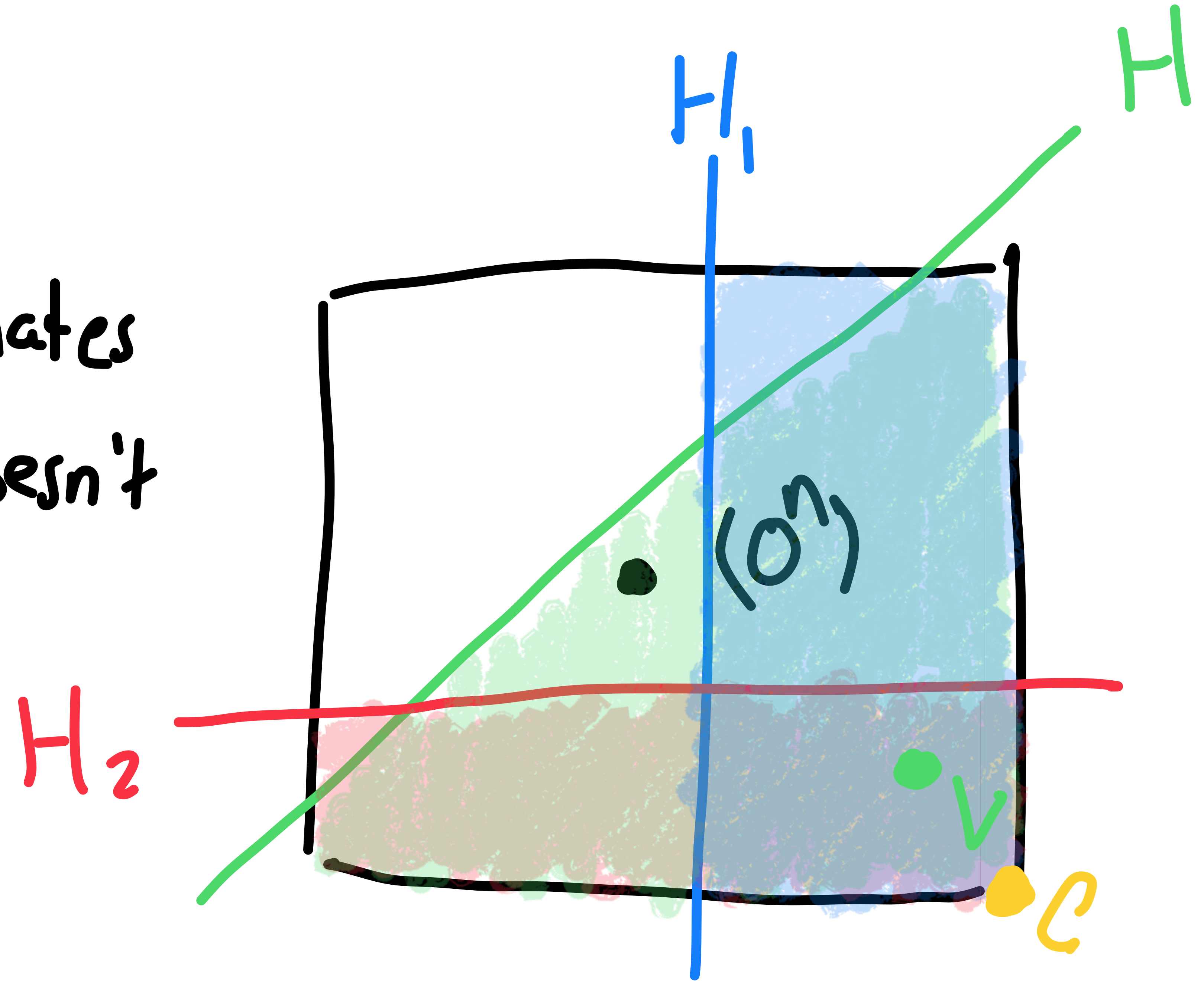

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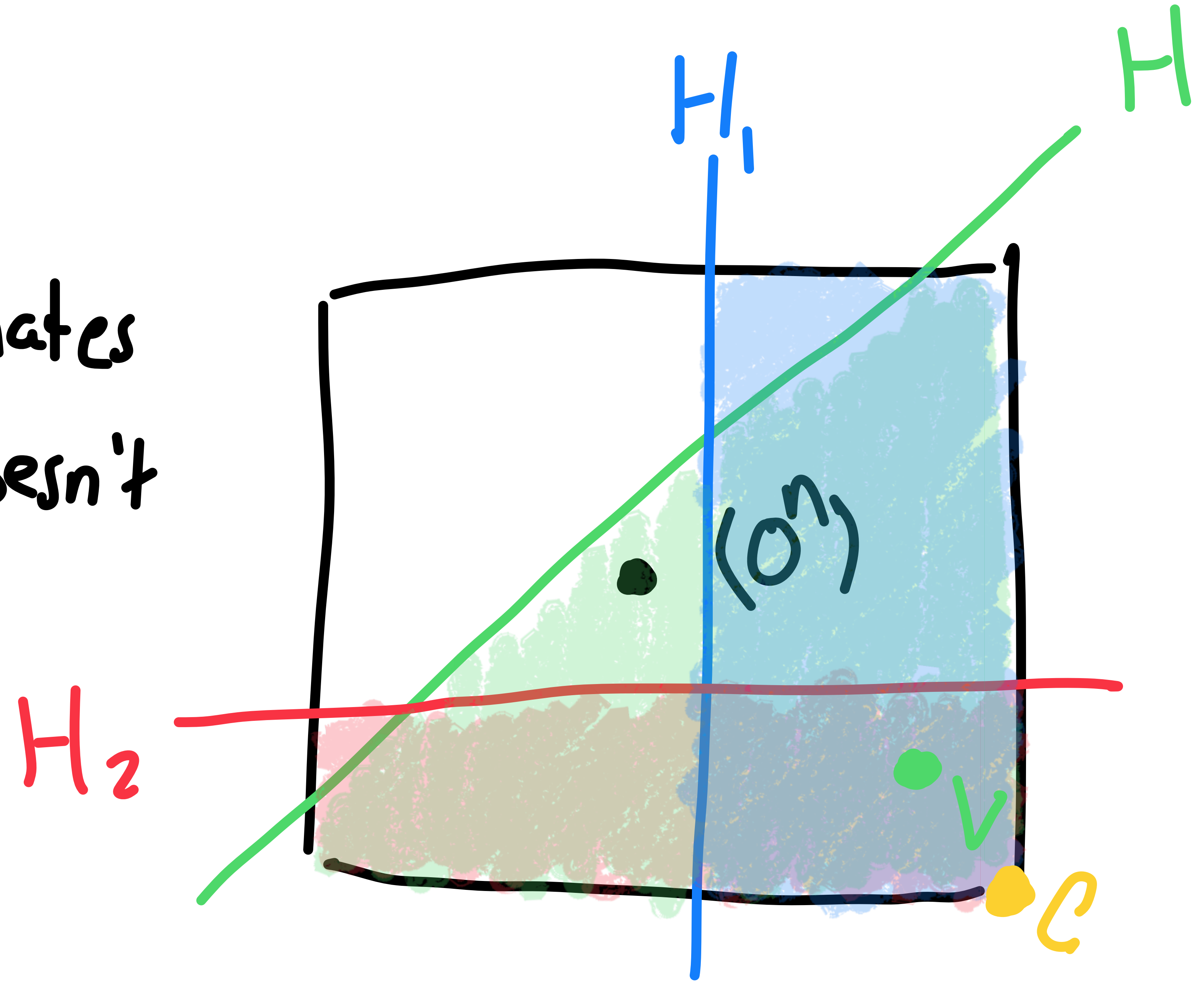

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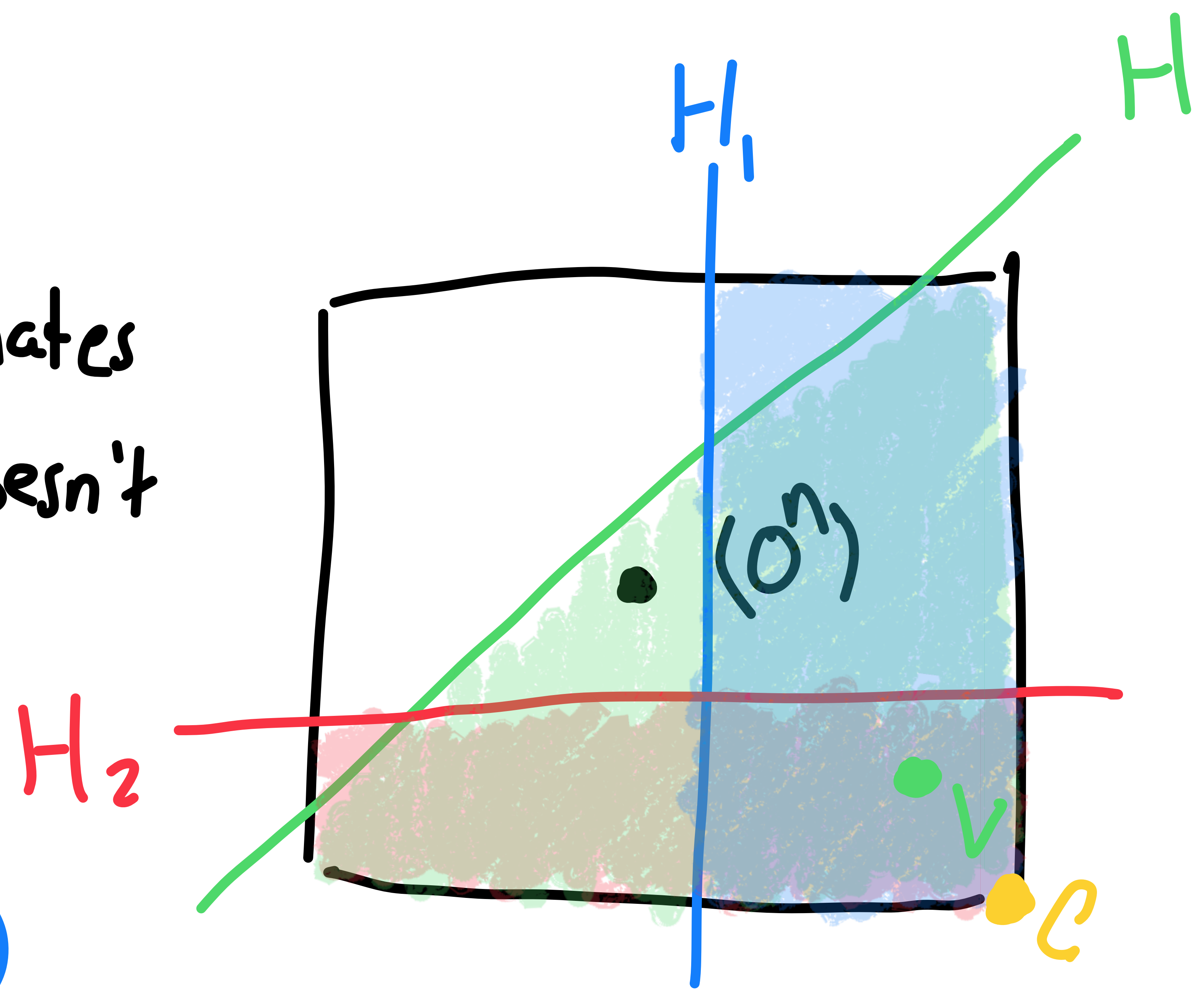

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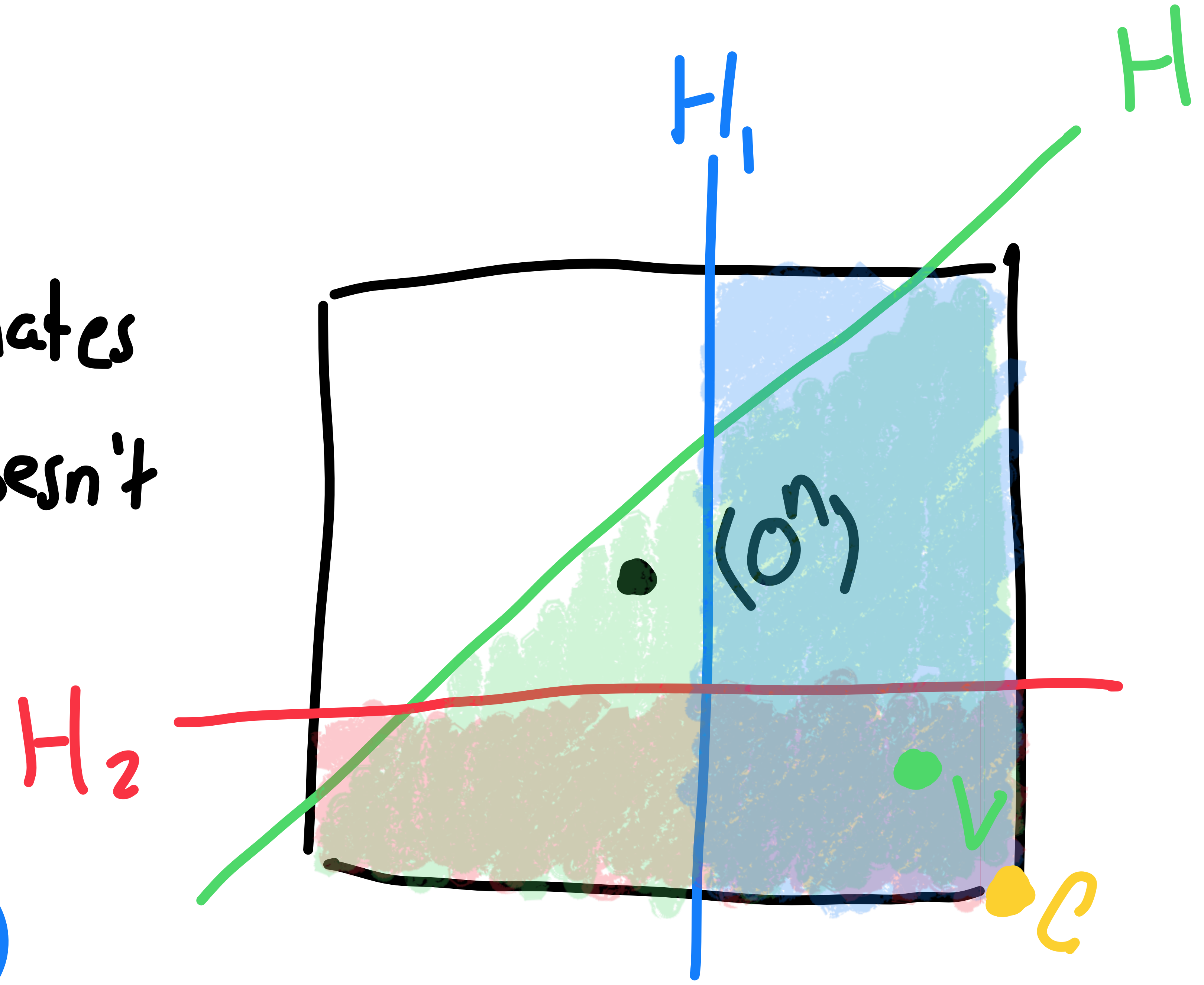

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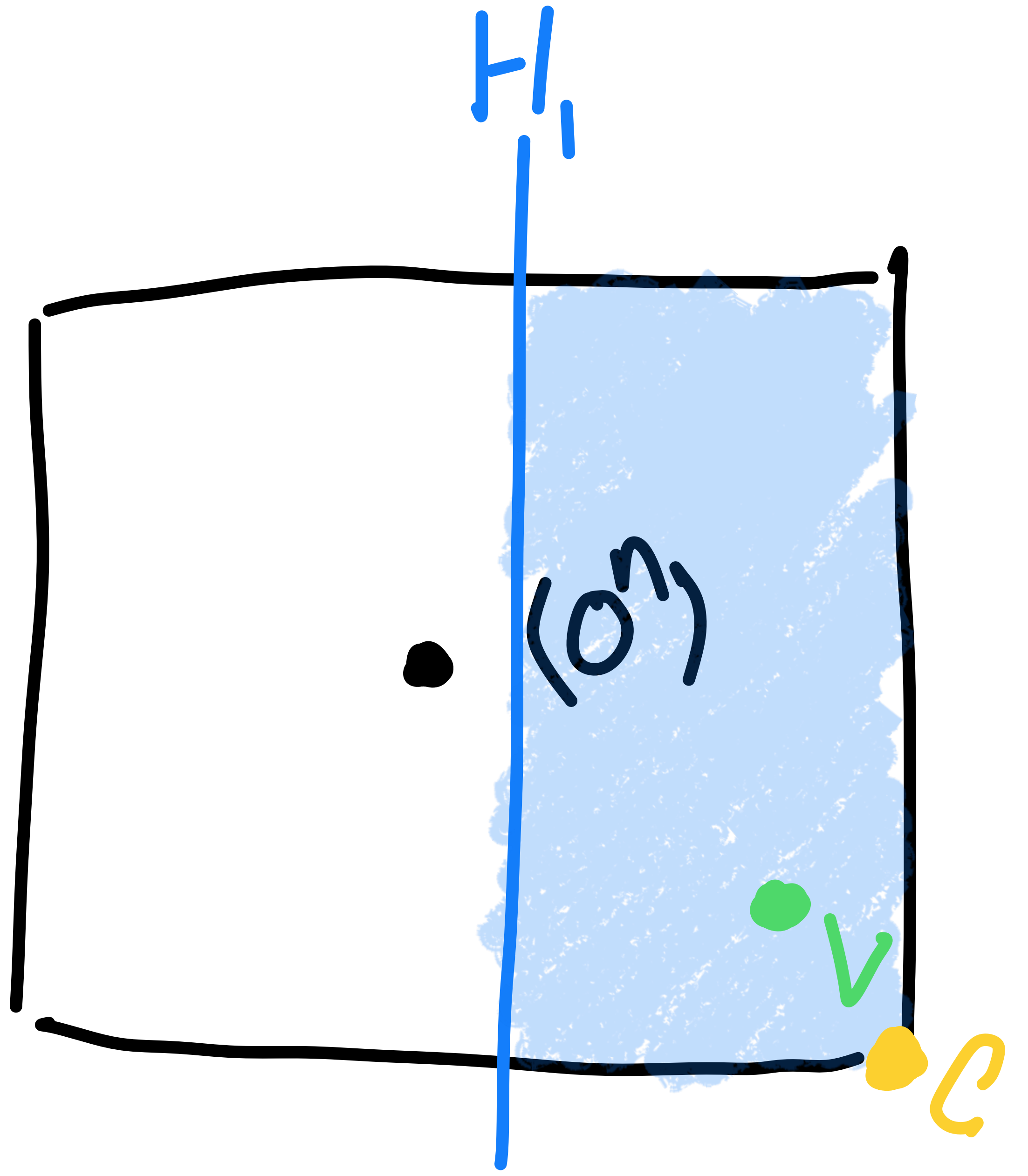

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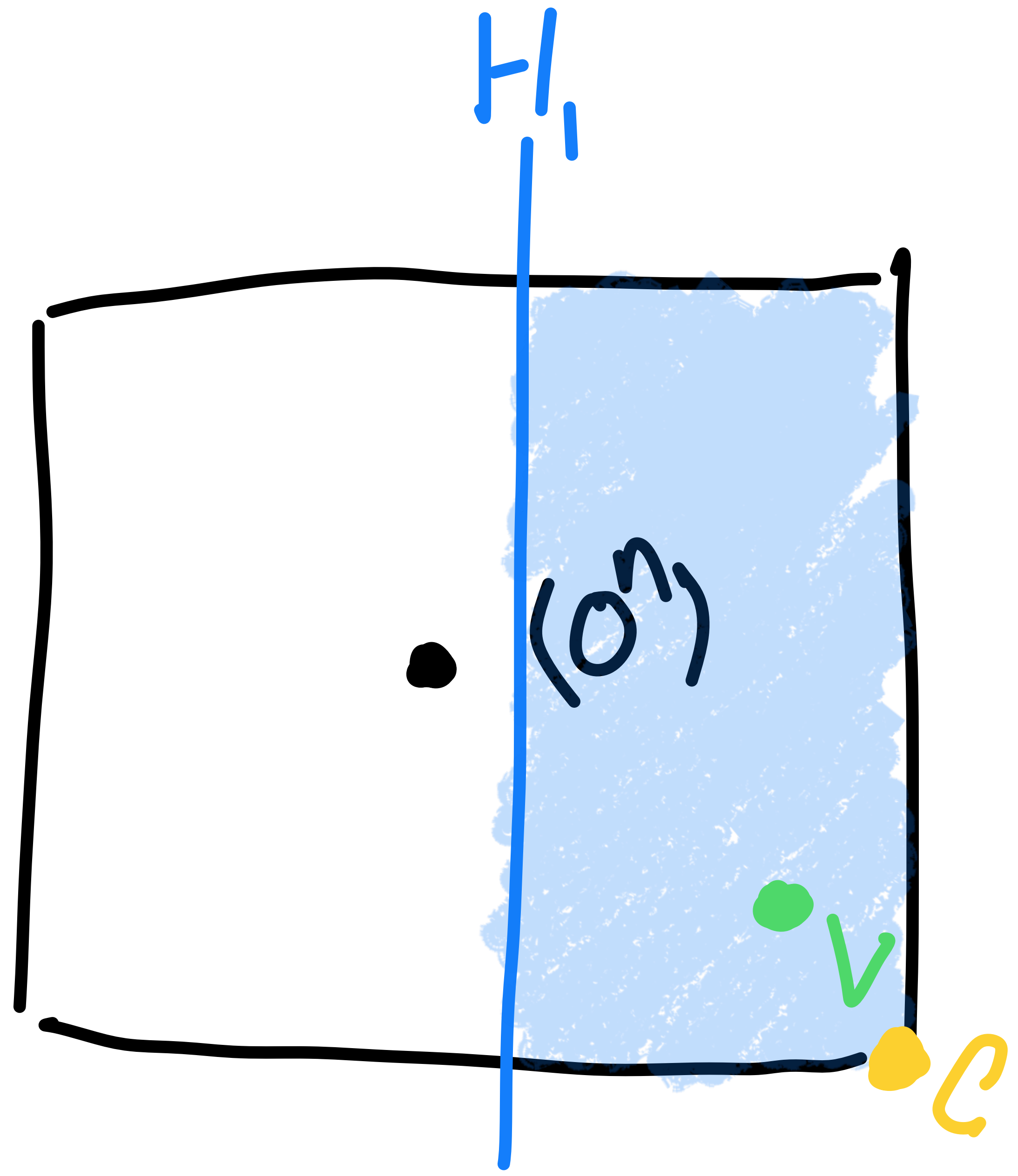

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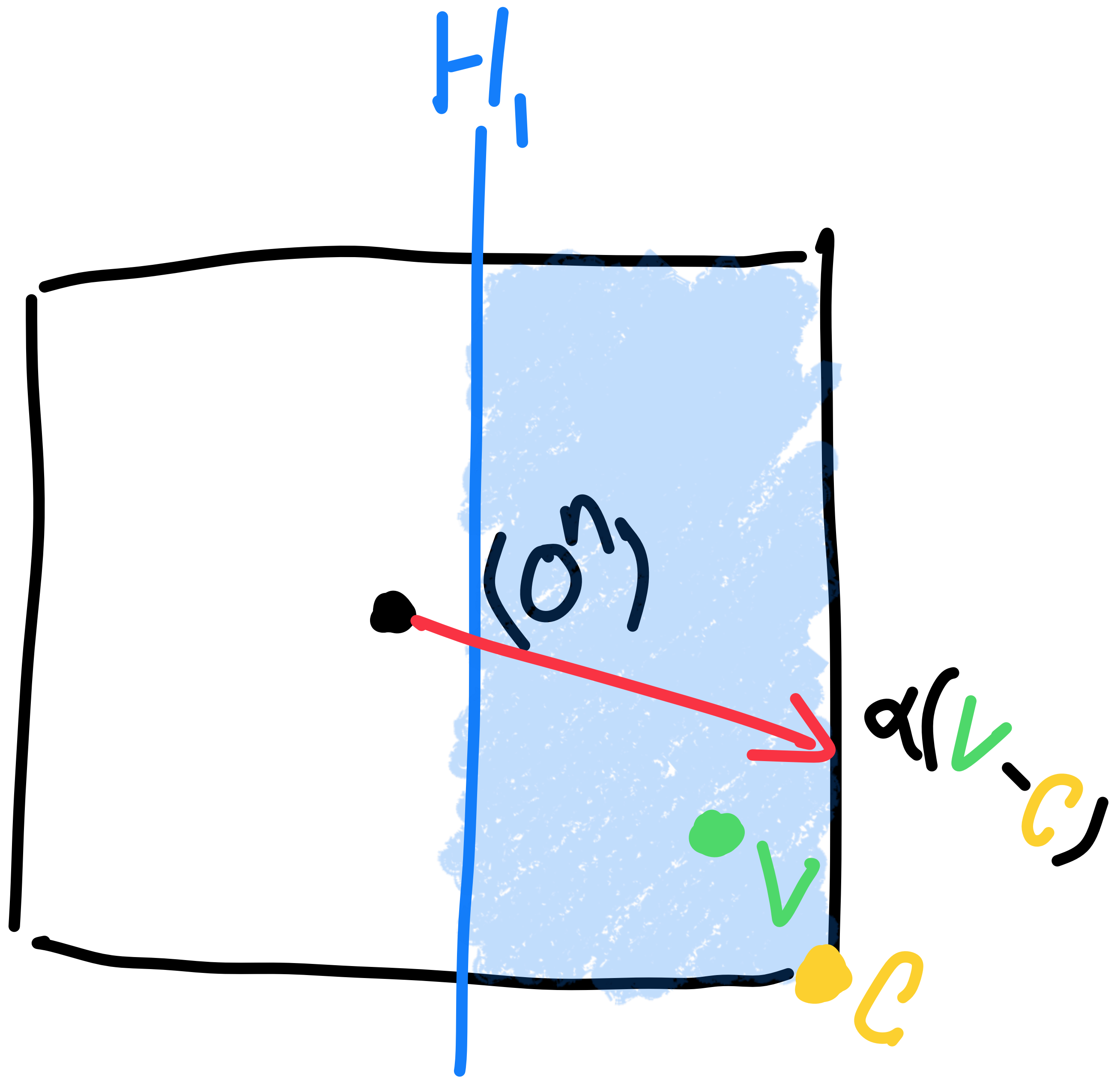

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  $$\rightarrow a_1(\alpha(c-v)) = \alpha a_1 c - \alpha a_1 v = \alpha a_1 c < b_1$$
- By monotonicity of H_1 we can round

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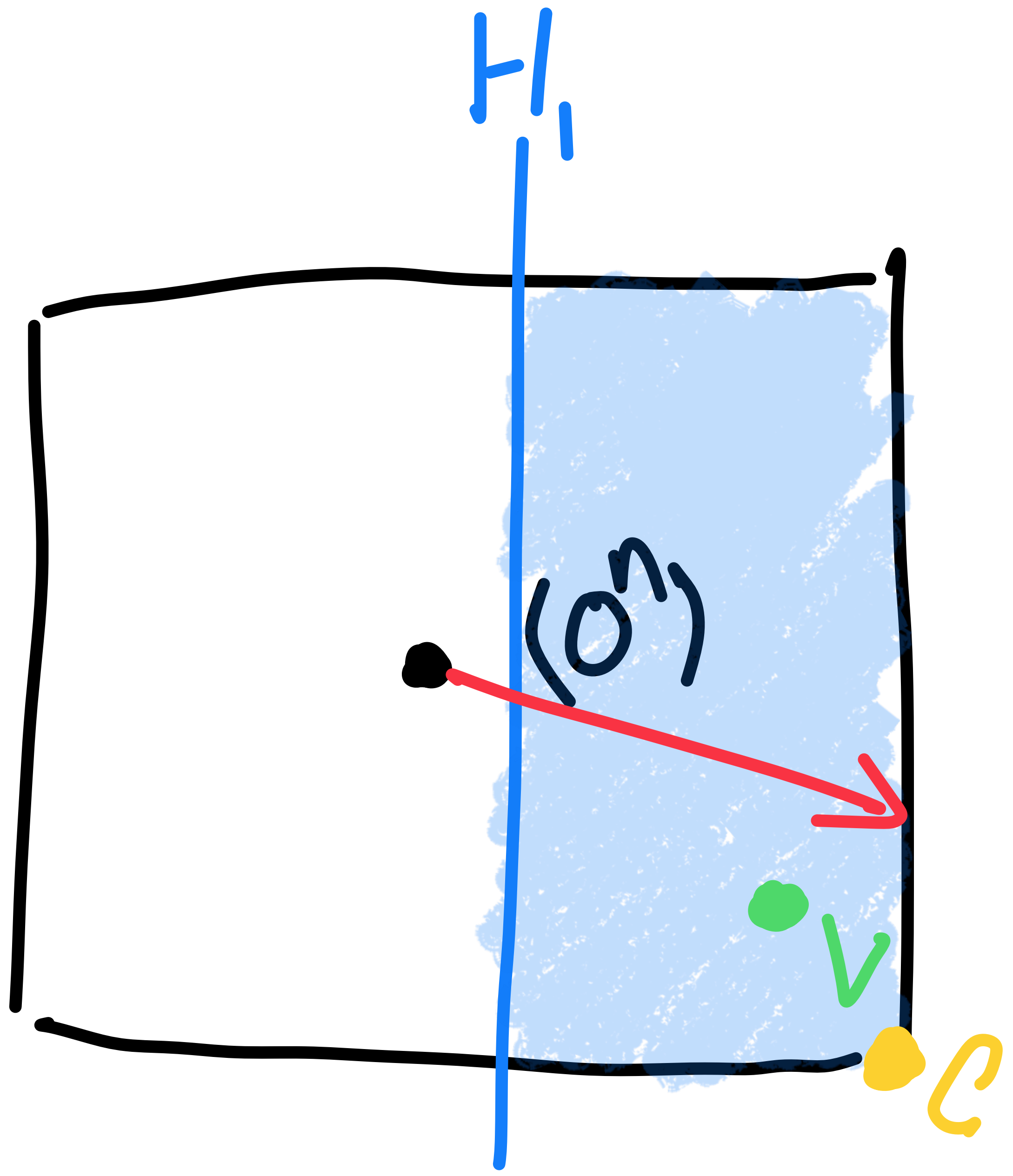

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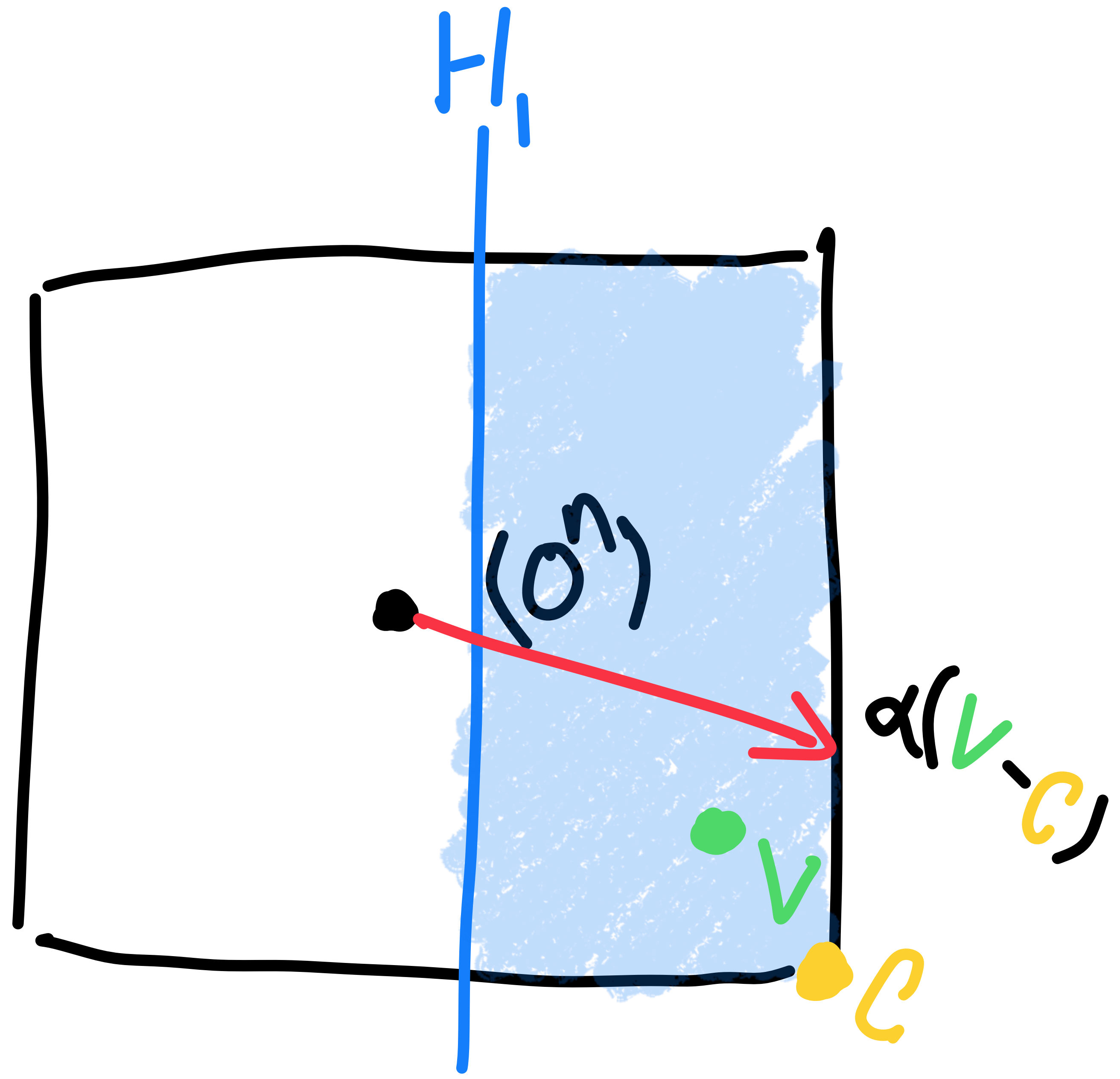

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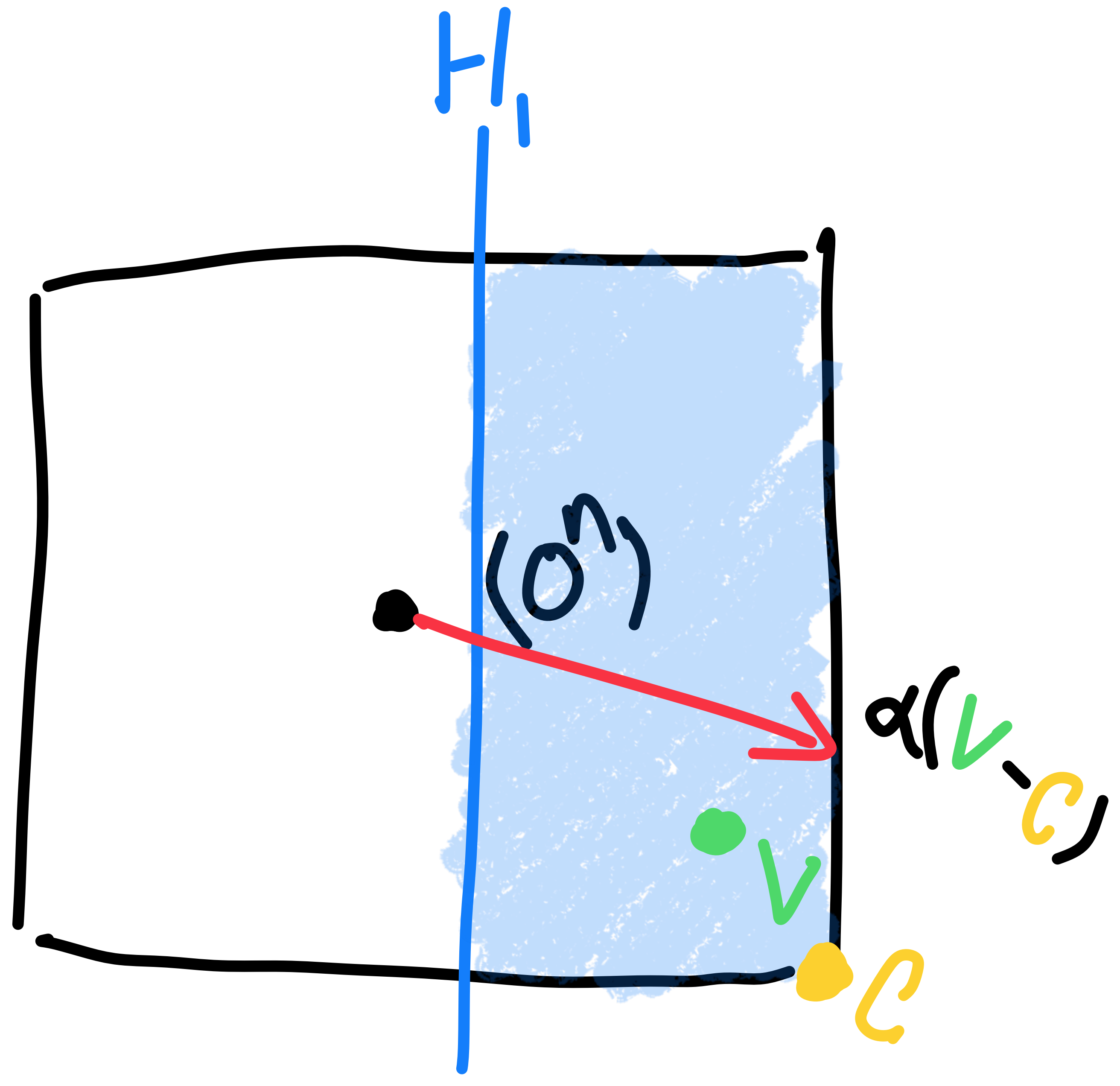

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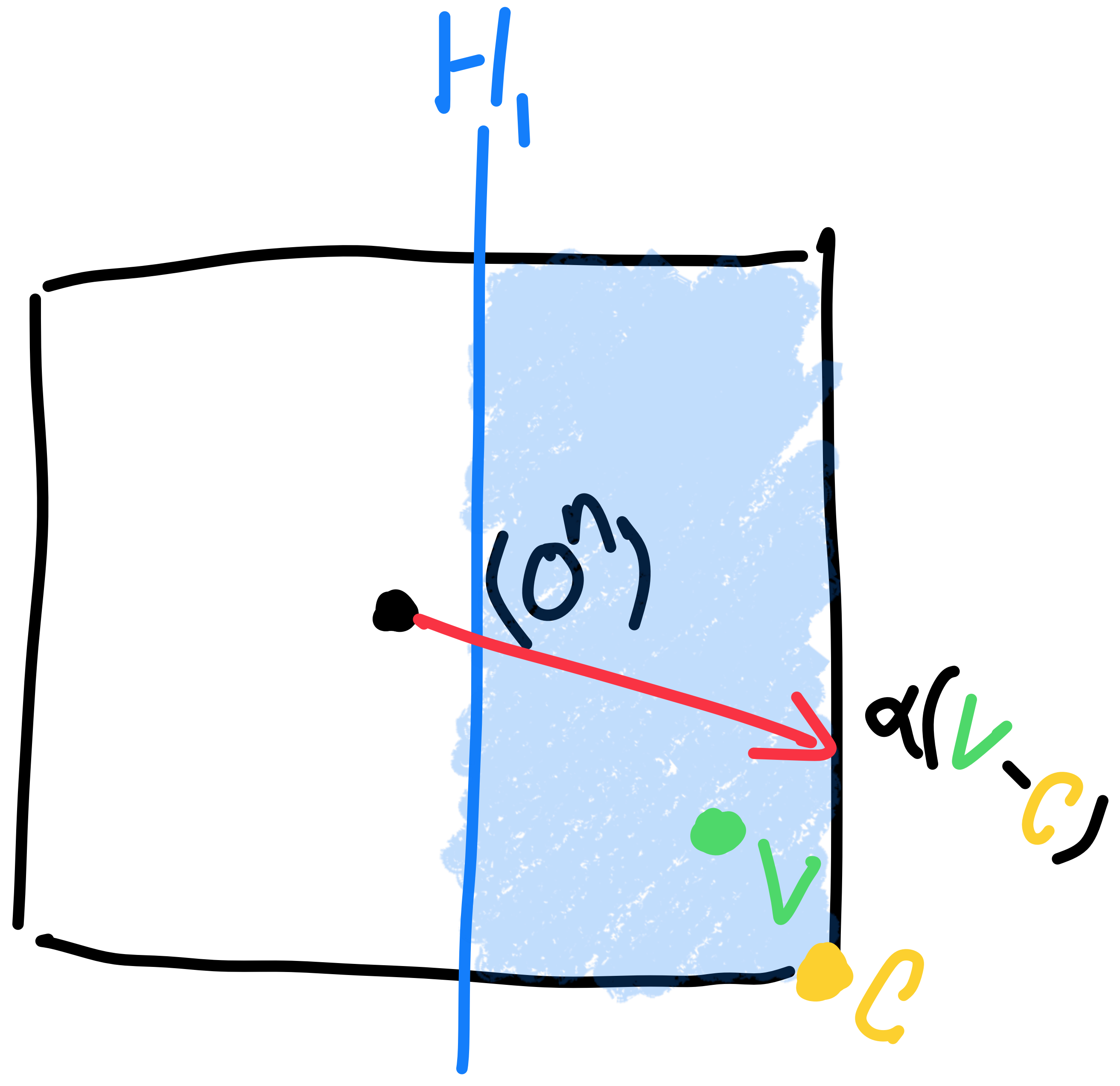

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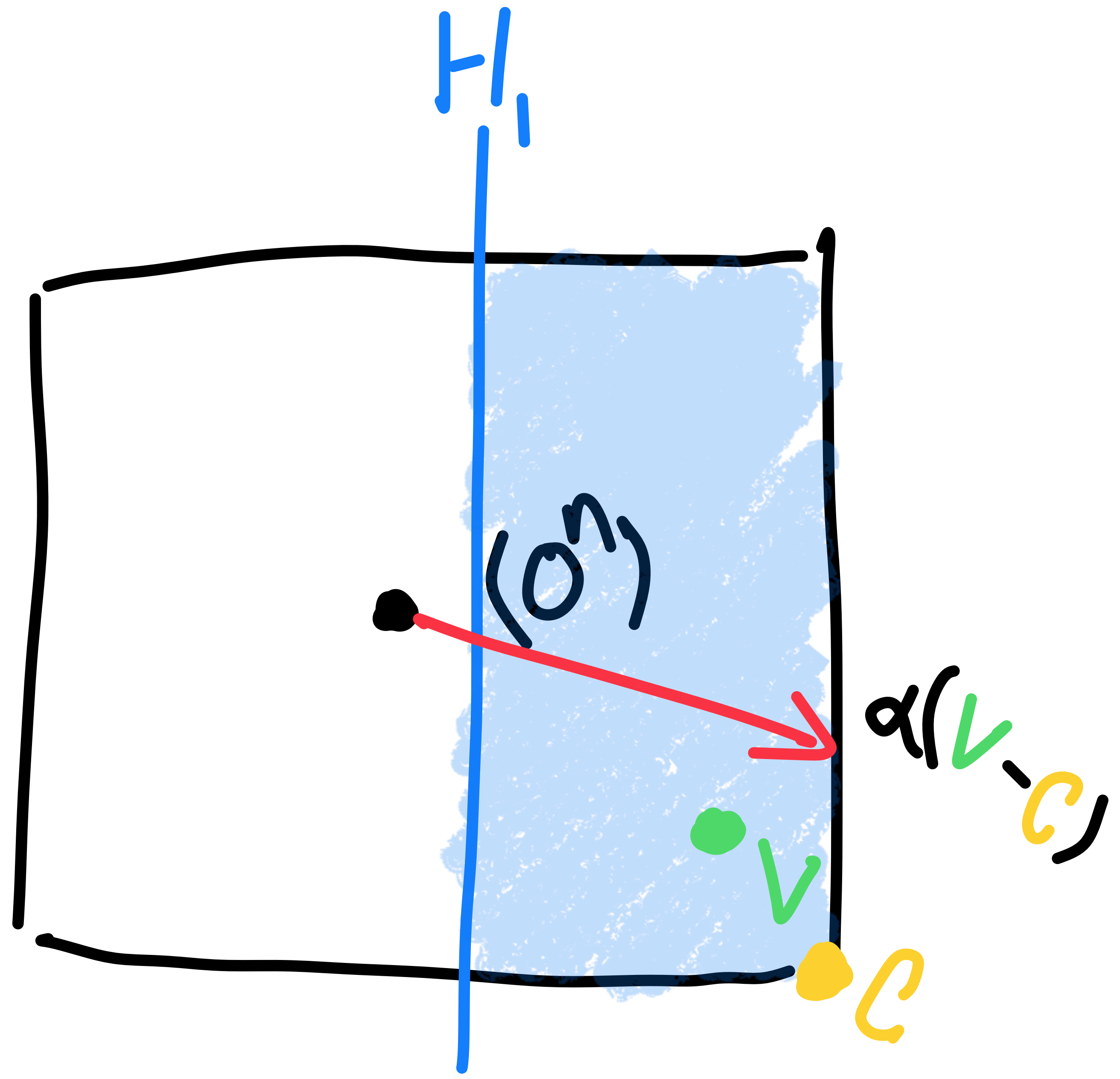

- Starting at $y = 0^n$ move in the direction of $c - v$ until one of the two non-zero coordinates becomes in $\{-1,1\}$. Let $\alpha(c-v)$ for $\alpha > 0$ be this point. Note: $\alpha > 1$ as c is the closest corner to v, so $\|c-v\|_\infty \leq 1$

  → $a_1(\alpha(c-v)) = \alpha a_1 c - \alpha a_1 v =$

Proof Idea of Technical Lemma

- Let $H_1 = a_1 x \geq b_1$, $H_2 = a_2 x > b_2$
- Find $v \in [-1,1]^n$ orthogonal to a_1, a_2 s.t. at most two coordinates of v are not $-1,1$ $\rightarrow$ moving in direction of v doesn't affect H_1, H_2 i.e. $a_1(x+v) = a_1 x$
- Let $c \in \{-1,1\}^n$ be closest to v
- Suppose $H(v)=0$ (otherwise set $v \leftarrow -v$, $c \leftarrow -c$)
  $\rightarrow$ $H_1(v)=0$ or $H_2(v)=0$ (assume H_1)
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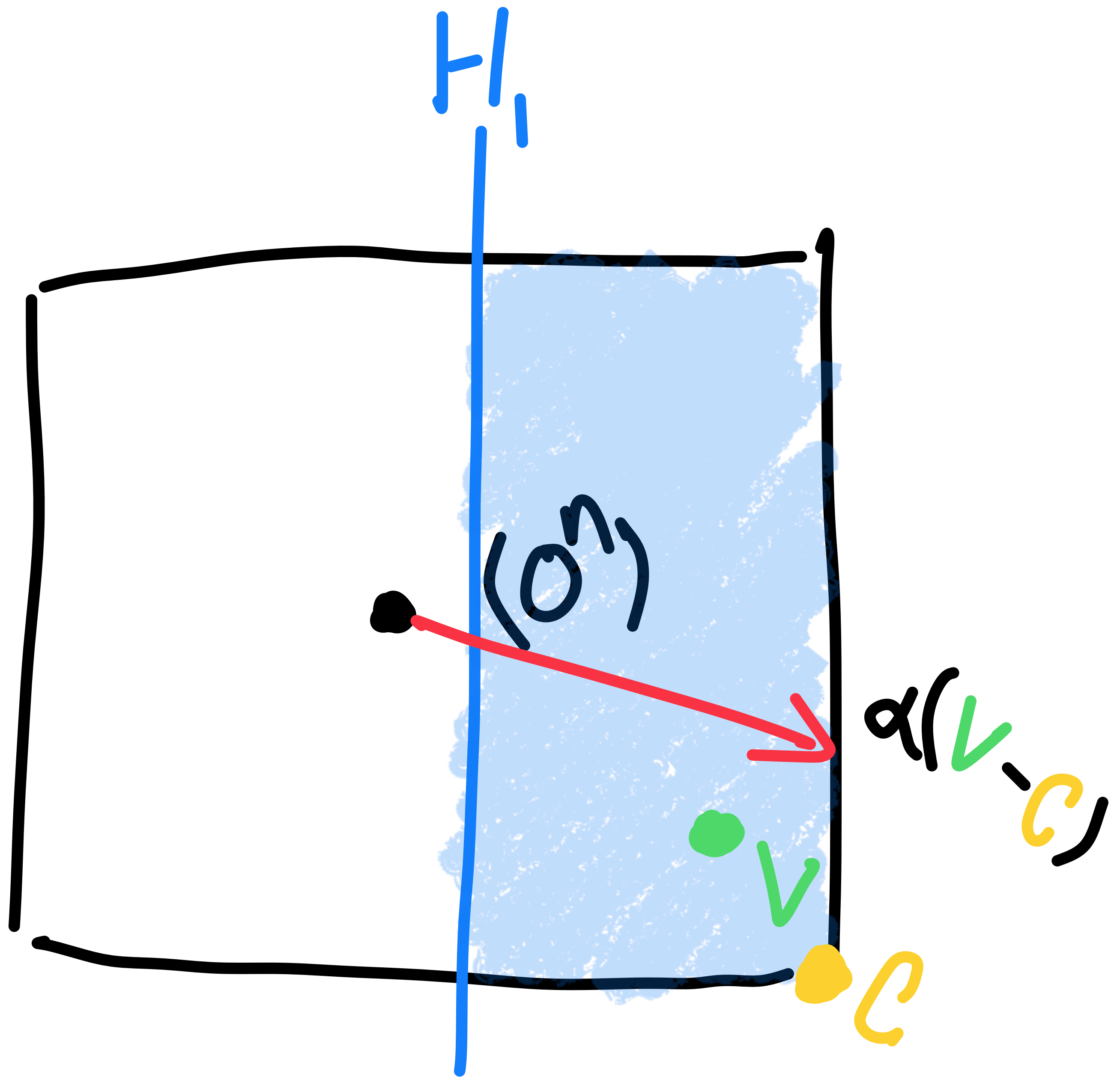

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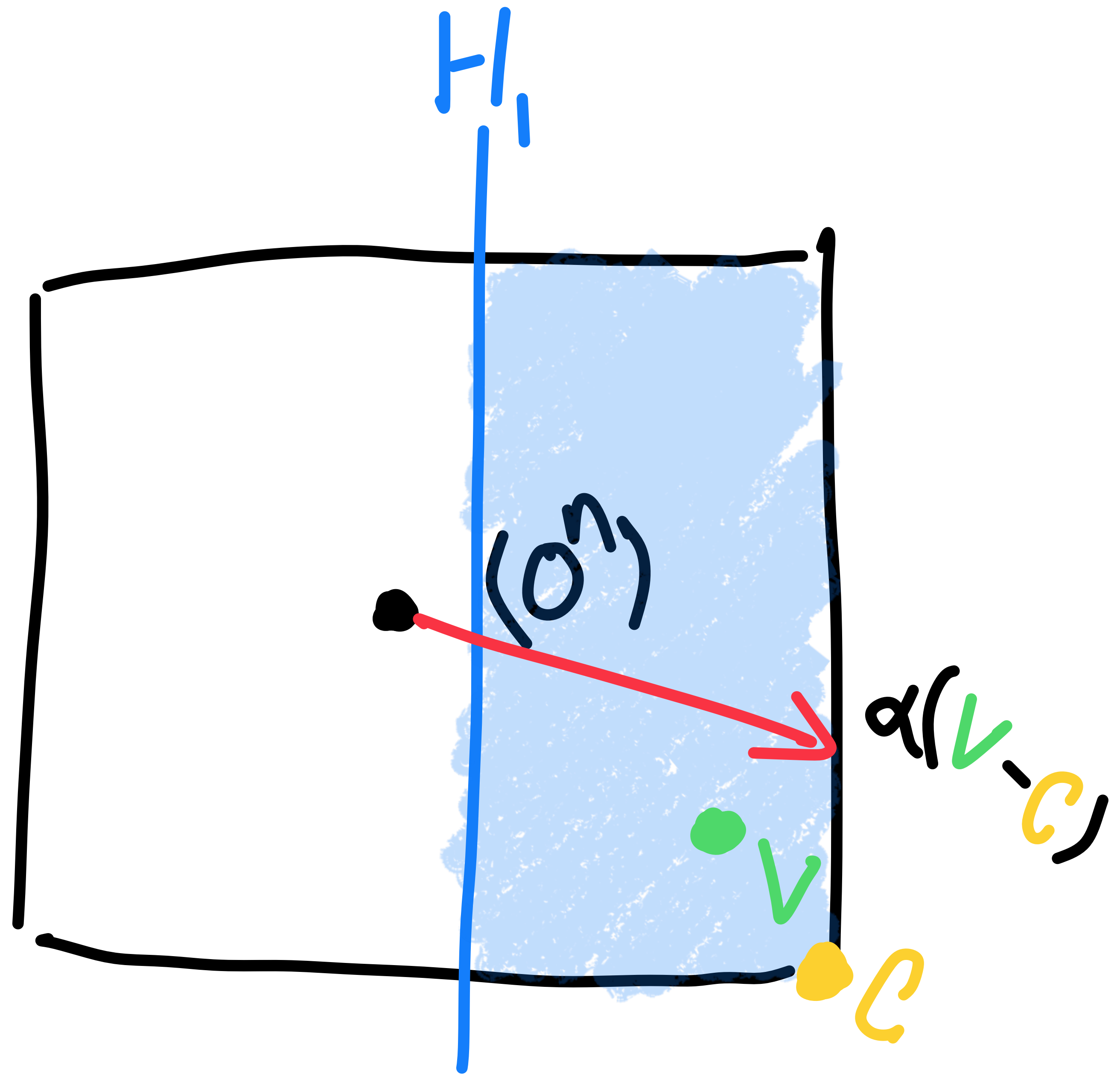
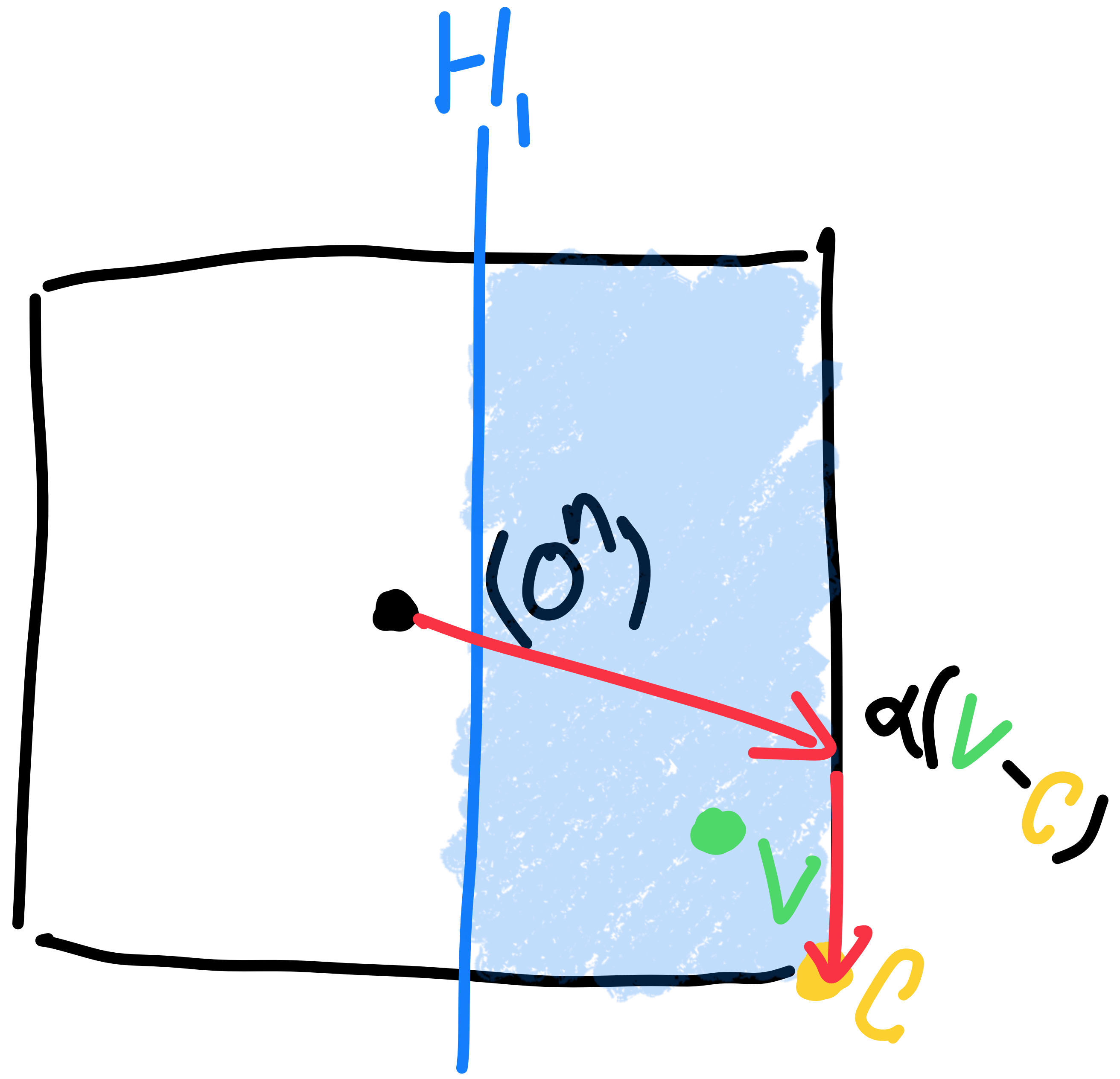
  $\rightarrow$ $a_1(\alpha(c-v)) = \alpha a_1 c - \alpha a_1 v = \alpha a_1 c < b_1$
- By monotonicity of H_1 we can round the remaining non-$\{-1,0,1\}$ coordinate to $\{-1,1\}$



Proof Idea of Technical Lemma



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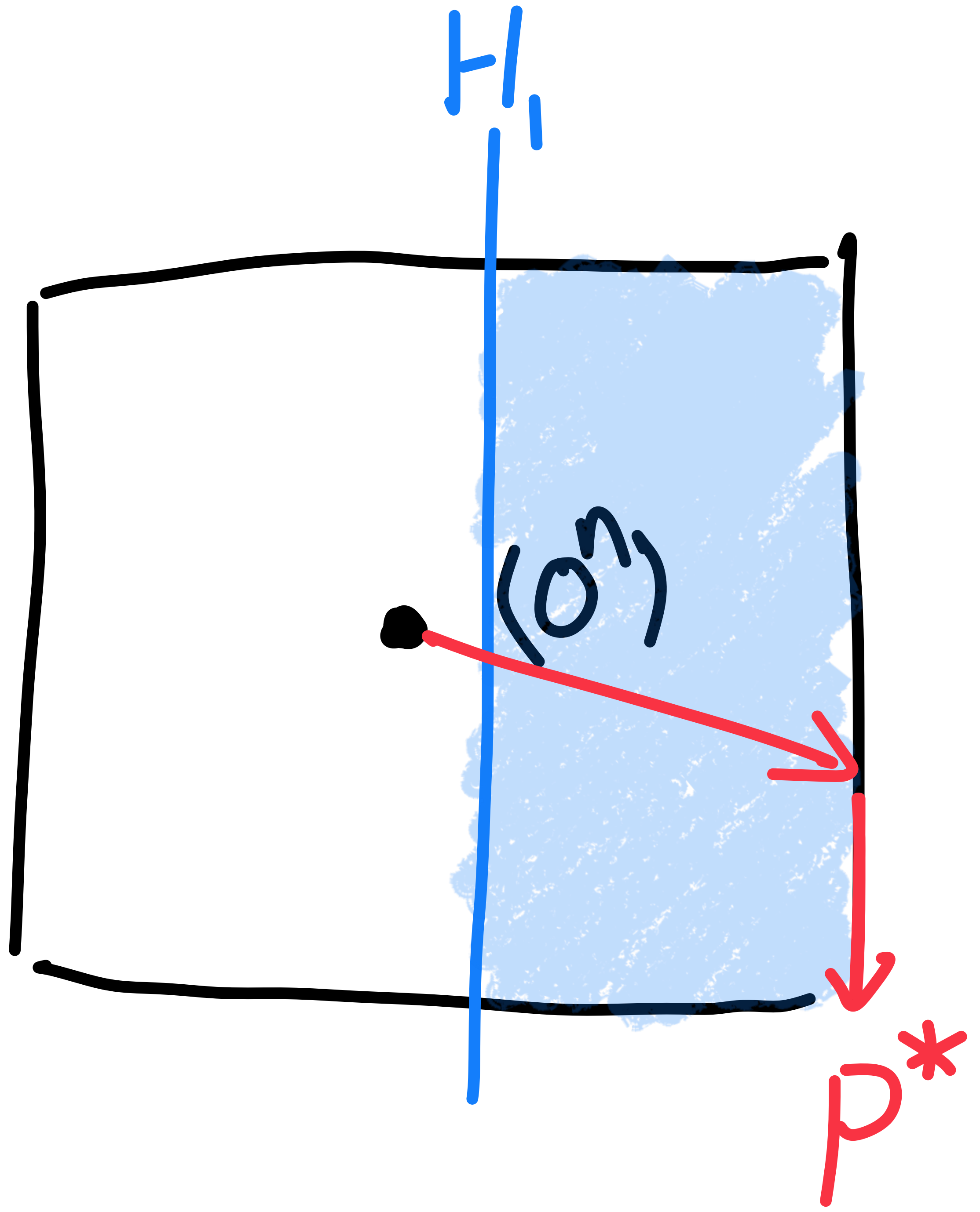
Proof Idea of Technical Lemma

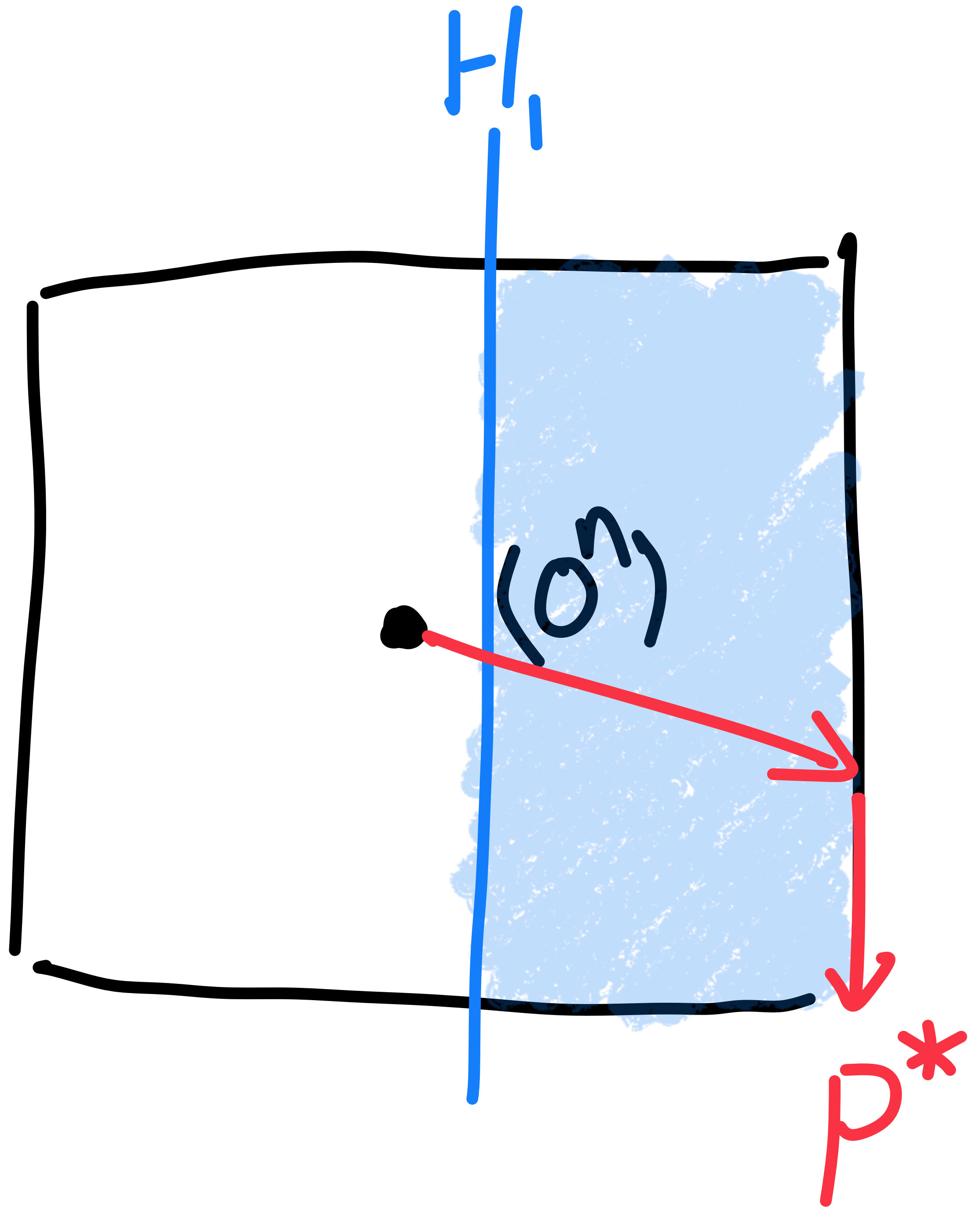
- ∴ We have found a point $p^* \in \{-1,0,1\}^n$ s.t.



Proof Idea of Technical Lemma

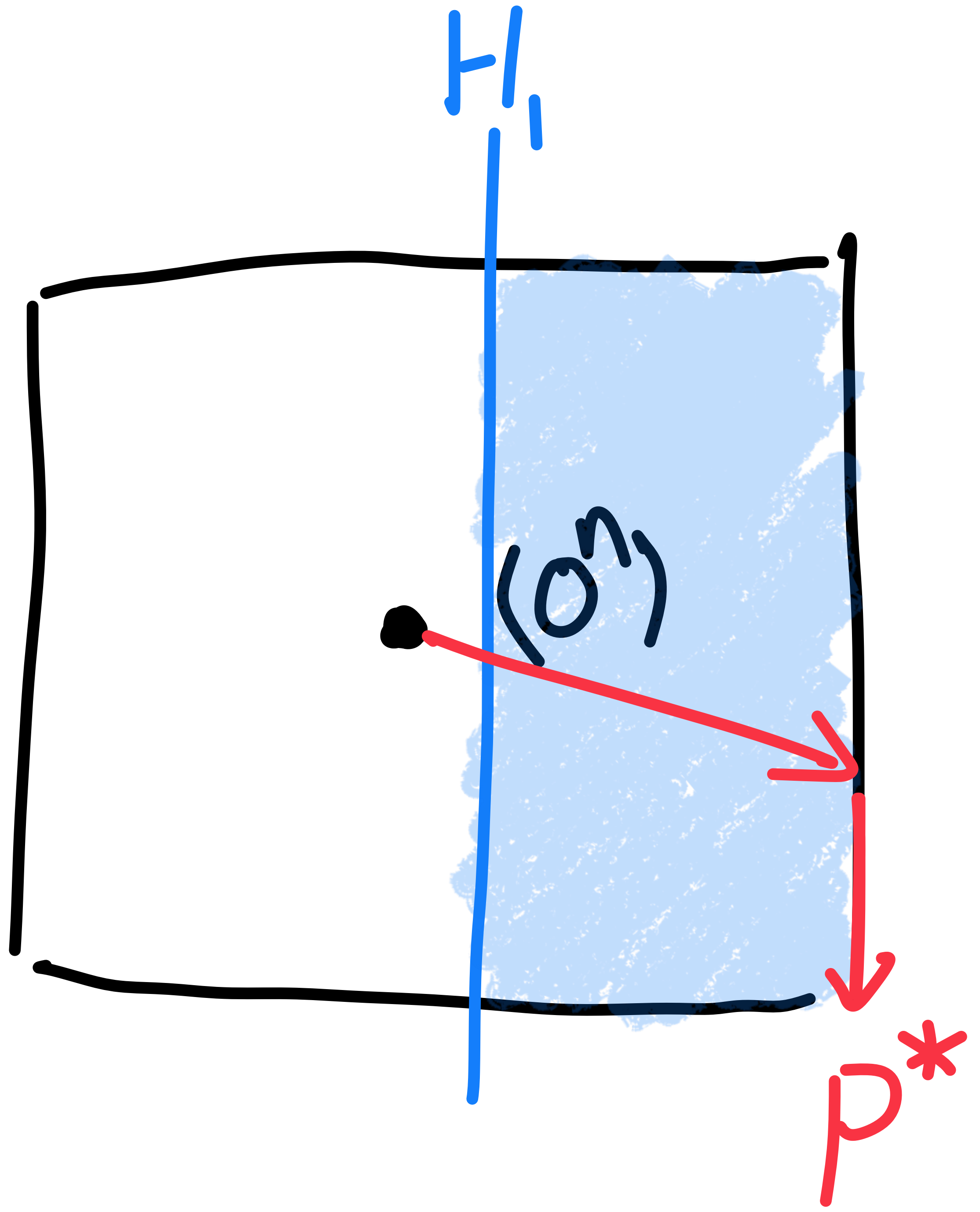
$\therefore$ We have found a point $p^* \in \{-1,0,1\}^n$ s.t.

- 2 coordinates are in $\{-1,1\}$



Proof Idea of Technical Lemma

- $\therefore$ We have found a point $p^* \in \{-1,0,1\}^n$ s.t.
  - 2 coordinates are in $\{-1,1\}$
  - $H_1(p^*) = 0$



Proof Idea of Technical Lemma

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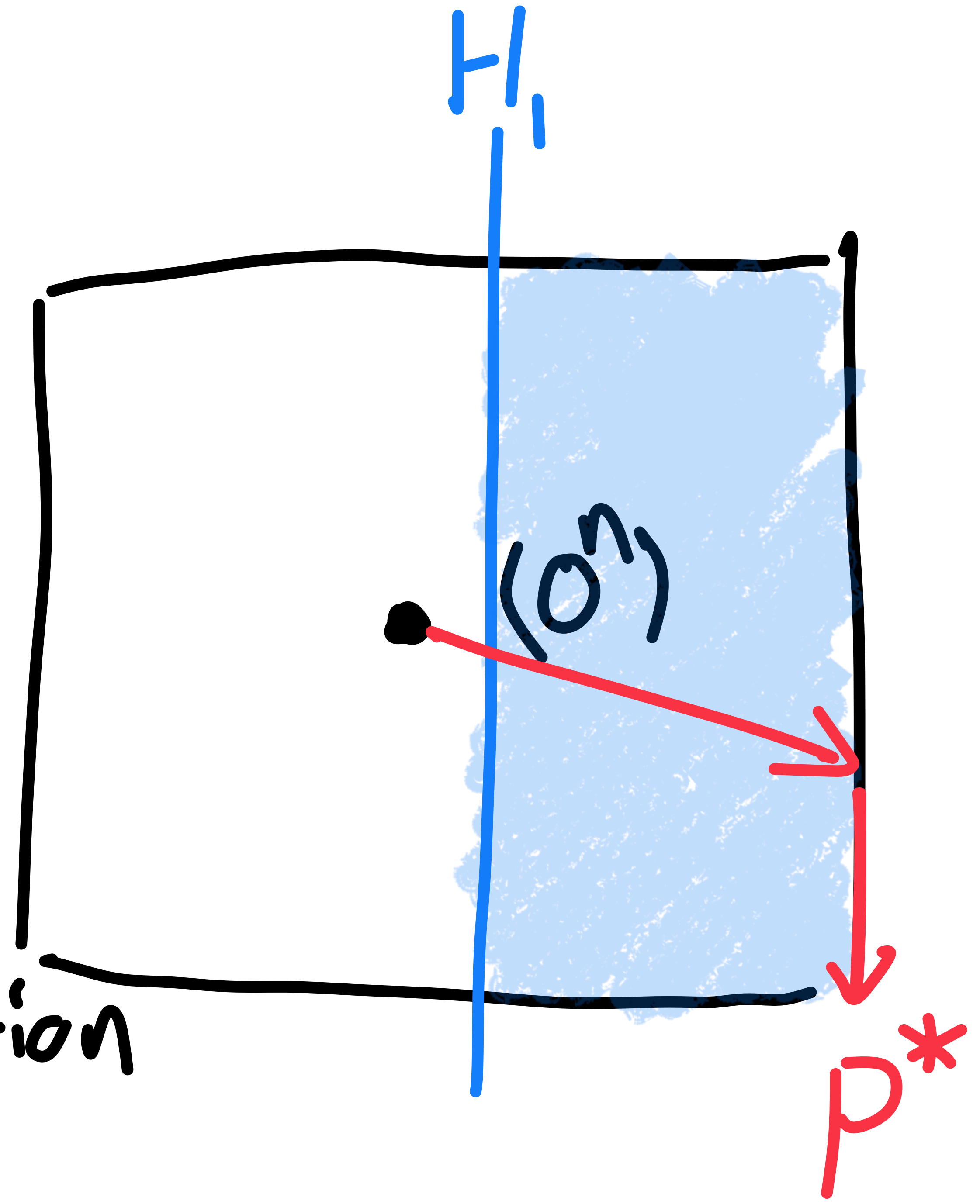
→ take the $\{-1,1\}$ coordinates to be our restriction p'

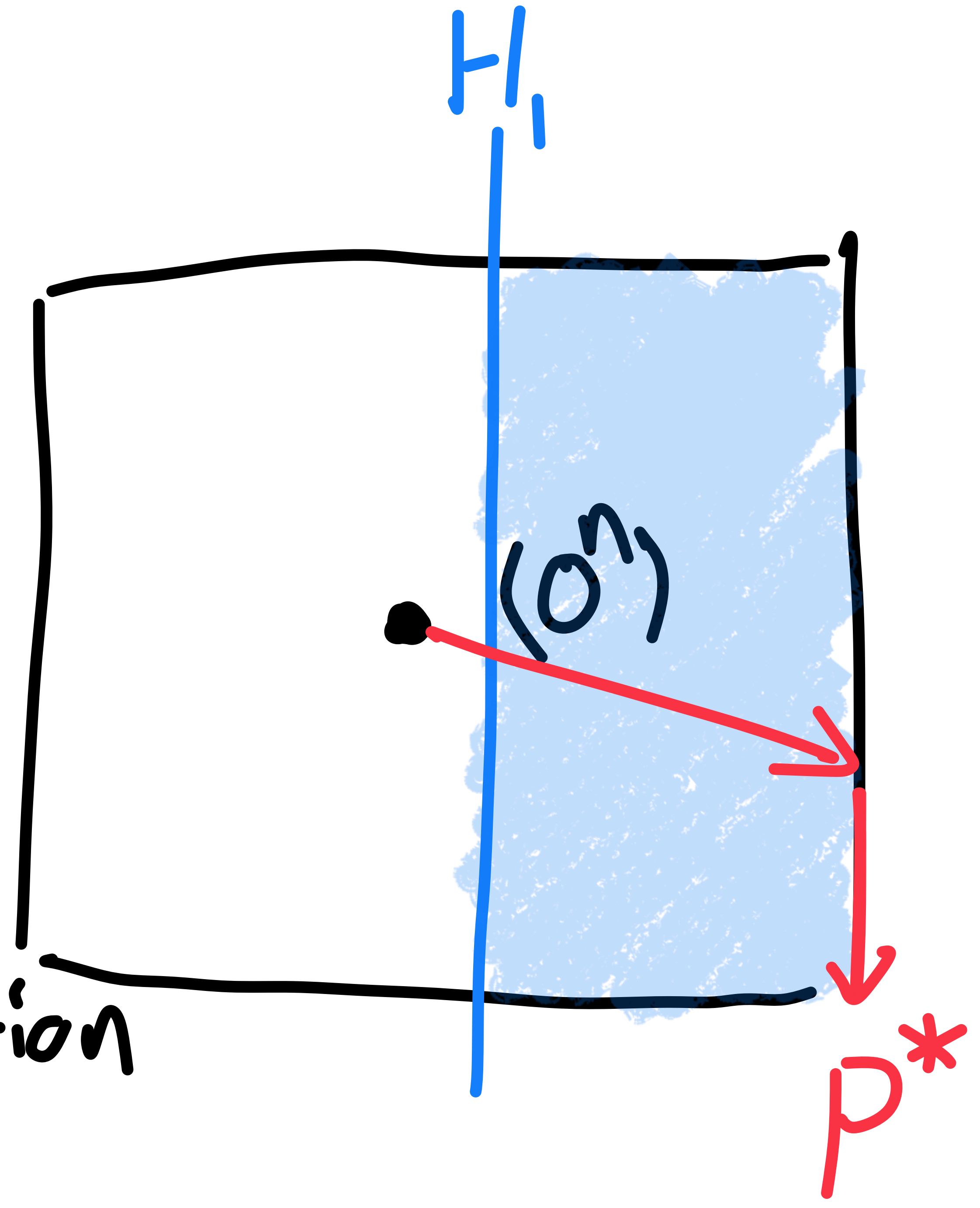


Proof Idea of Technical Lemma

$\therefore$ We have found a point $p^* \in \{-1,0,1\}^n$ s.t.

- 2 coordinates are in $\{-1,1\}$
- $H_1(p^*) = 0$

$\rightarrow$ take the $\{-1,1\}$ coordinates to be our restriction $p' \longrightarrow H_1\restriction p'$ is good.



Open Problems

- Can CP p-simulate SP*?
- Can CP (quasipolynomially) simulate SP?
- Can CP* (quasipolynomially) simulate SP*?
- Can SP or CP simulate dag-like SP (R(CP))?
  - → CP cannot p-simulate R(CP) [ABE02]
- Can treelike CP refute systems of $\mathbb{F}_2$ linear equations?